CONSEQUENCES OF THE PENTAGON PROPERTY

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1. Introduction

Suppose we have a set of points, $\mathcal{S}$, in the plane such that any convex pentagon whose vertices are in $\mathcal{S}$ contains another point of $\mathcal{S}$ in its interior. What can we then say about the number of points of $\mathcal{S}$ that must lie in the interior of a convex n-gon whose vertices are in $\mathcal{S}$?

This question was suggested by the fact that the lattice points in the plane have this property. (A lattice point is a point with integer coordinates.) See, for example, the Lattice Pentagon Theorem in [8].

Let $g(n)$ denote the largest integer such that a convex lattice n-gon must necessarily contain $g(n)$ lattice points.

Several authors ([7], [8], [13]) have investigated the properties of $g(n)$. They showed that $g(6) = 1$, $g(7) = 4$, $g(8) = 4$, $g(9) = 7$, $g(10) = 10$, $g(11) \geq 11$, $g(12) = 19$, $g(13) = 27$, $g(14) = 34$, $g(15) \geq 43$, and $g(16) = 52$. Coleman [7] used number theory and Simpson [13] used area during their investigations. Rabinowitz [8] used other properties of lattice points as well. Such mechanisms may not be needed. It is possible that the results can be proven purely by combinatorial methods. Instead of working with lattice points, one can work with any set of points in the plane that satisfies the Pentagon Property:

The Pentagon Property. A set, $\mathcal{S}$, of points in the plane is said to have the *Pentagon Property*, if any convex pentagon whose vertices are in $\mathcal{S}$ must also contain a point of $\mathcal{S}$ in its interior.

From now on, we will consider a fixed set of points, $\mathcal{S}$, that satisfies the Pentagon Property. We will call these points *special points*. We don't want these points to be dense, so we will also add the technical condition that any bounded subset of the plane contains only a finite number of points in $\mathcal{S}$. We only need this extra condition in the proof of Theorem 1. A convex polygon whose vertices are all special will be called a *special polygon*.

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We will now prove some consequences of the Pentagon Property. These proofs will in some cases be the same as corresponding proofs in [8], but in many cases, we give simpler proofs.

If K is a special polygon, then we let $g(K)$ denote the number of special points in the interior of K, we let $v(K)$ denote the number of vertices of K, and we let $b(K)$ denote the number of special points on the boundary of K. A special n-gon is a special polygon with n vertices.

The notation "$ABCDE \Rightarrow X$" shall mean "Polygon $ABCDE$ is a special pentagon and hence contains a special point, X, in its interior by the Pentagon Property".

Theorem 1 (The Quadrangular Segment Theorem). Let $ABCDE$ be a special pentagon. Then there is a special point inside quadrilateral $ABCD$ or on segment AD.

Proof. Let P be the special point inside $\triangle ADE$ that is closest to segment AD. (If there is a special point in $\triangle ADE$, there must be a closest one because of the condition that there can only be a finite number of points of $\mathcal{S}$ in $\triangle ADE$.)

If there is no such special point inside $\triangle ADE$, then let P be E. See figure 1. Then $ABCDP \Rightarrow Q$. Point Q cannot lie inside $\triangle ADP$ for then it would be closer to AD than P, contradicting the definition of P. Thus Q lies on AD or inside quadrilateral $ABCD$ as claimed.

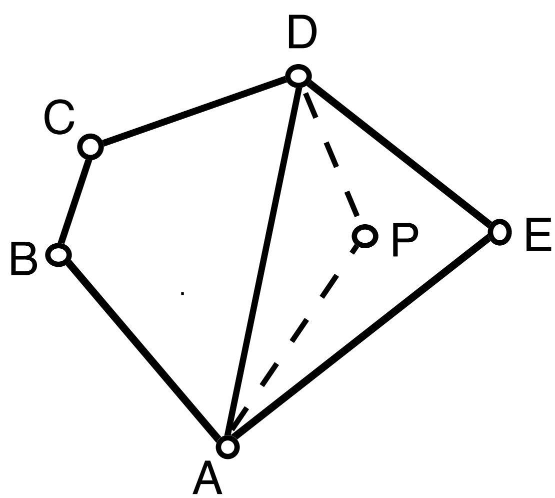

figure 1

2. The Hexagon

Theorem 2. If $v(K) = 6$, then $g(K) \geq 1$.

Proof. Let $ABCDEF$ be a special hexgon. See figure 2. Then $ABCDE \Rightarrow G$.

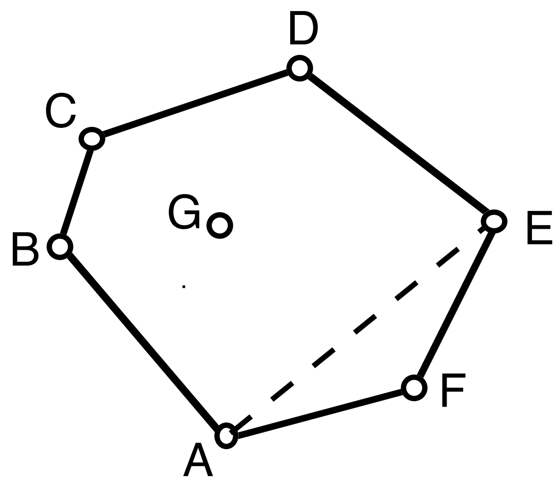

figure 2

3. The Heptagon

Theorem 3. If $v(K) = 7$, then $g(K) \geq 4$.

Proof. Let $ABCDEFG$ be a special 7-gon. Then $ABCDE \Rightarrow X$ (figure 3a) and $AXEFG \Rightarrow Y$ (figure 3b). The line XY can pass through at most two of the vertices of the 7-gon. The remaining 5 vertices lie in the two open halfplanes bounded by XY. By the pigeonhole principle, one of these open halfplanes must contain at least 3 of the vertices. Suppose these three vertices are D, E, and F (figure 3c). Then $XDEFY \Rightarrow Z$. Note further that the three interior special points we have found are not collinear.

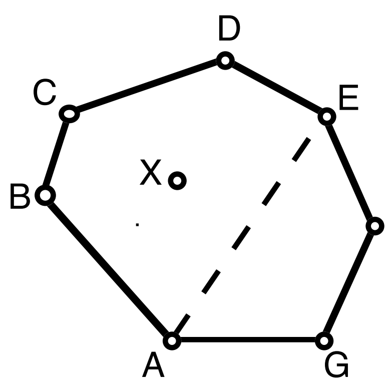

figure 3a

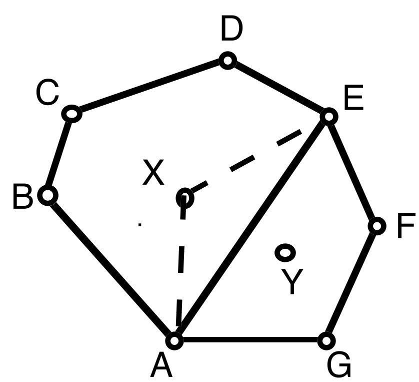

figure 3b

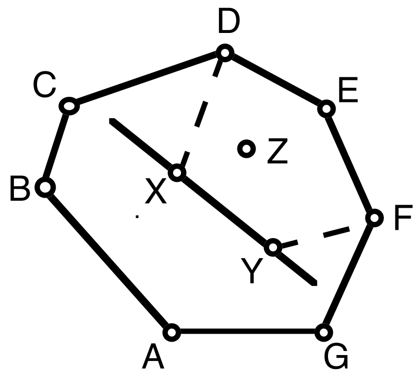

figure 3c

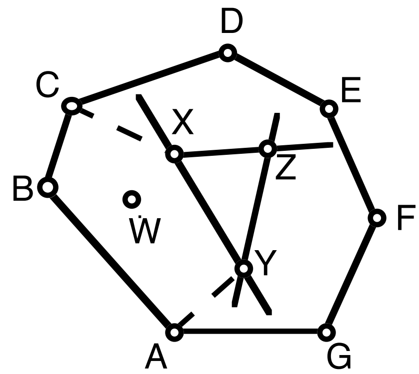

figure 3d

Now consider the three open halfplanes bounded by the sides of triangle XYZ but which do not contain the interior of triangle XYZ. These three open halfplanes completely cover the exterior of $\triangle XYZ$. The 7 vertices of the polygon lie in the exterior of $\triangle XYZ$, so by the pigeonhole principle, some three of them, say A, B, C, lie in the same halfplane, say the one bounded by XY (figure 3d). Then $XYABC \Rightarrow W$.

4. The Interior Hull

Definition. Let K be a convex body in the plane. Then $H(K)$ is the boundary of the convex hull of the special points in the interior of K. $H(K)$ is called the *interior hull* of K. This will frequently be denoted by just H, if K is fixed.

In other words, H is the largest special polygon whose vertices are in the interior of K. Note, however, that H might degenerate into a line segment, a point, or even the null set.

In this section, we will investigate the relationship between a special polygon, K, and its interior hull.

Definitions. Let H be the interior hull of a special polygon. If AB is an edge of H, then $h(AB)$ denotes the open halfplane bounded by AB that is exterior to H, and $h^*(AB)$ denotes the other open halfplane.

Theorem 4. (The 3-vertex Restriction). Let K be a special polygon and let H be the interior hull of K. Let XY be an edge of H. Then $h(XY)$ contains at most two vertices of K.

Proof. Suppose the open halfplane, $h(XY)$, contains 3 vertices of K, say A, B, and C. See figure 4. This would be a contradiction because $ABCYX \Rightarrow P$.

Theorem 5 (The Interior Hull Vertex Inequality). Let K be a special polygon with interior hull H. If $v(K) \geq 7$, then $v(H) \geq \lceil \frac{1}{2} v(K) \rceil$.

Proof. Since $v \geq 7$, we saw in the proof of Theorem 3 that the interior hull forms a polygon. The number of edges of this polygon is $v(H)$. By the 3-vertex Restriction, for each edge AB of this polygon, $h(AB)$ contains at most two vertices of K. These halfplanes cover all of the vertices of K. Thus the total number of vertices of K is at most $2v(H)$. Hence $v(K) \leq 2v(H)$.

Theorem 6 (The Interior Hull Boundary Inequality). Let K be a special polygon with interior hull H. If $v(K) \geq 9$, then $b(H) \geq \lceil \frac{2}{3} v(K) \rceil$.

Proof. By the Interior Hull Vertex Inequality, we see that $v(K) \geq 9$ implies $v(H) \geq 5$. Then $g(H) \geq 1$ by the Pentagon Property. Thus, there is a special point, P, in the interior of H.

Draw rays from P to each of the special points on the boundary of H. Note that there will be no special points on the boundary of H between any two adjacent rays. Also, the angle between two adjacent rays will always be smaller than π. We have thus divided H into $b(H)$ wedges.

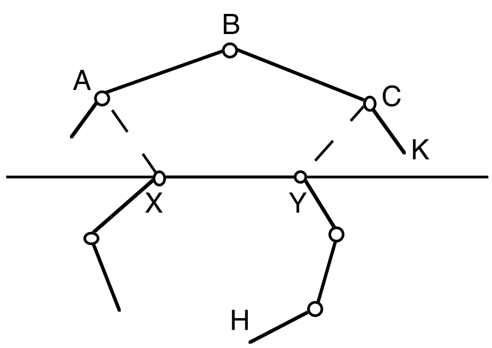

figure 4

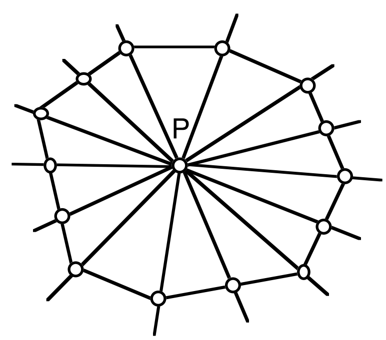

figure 5

For the purposes of this proof, consider an edge of a polygon to be the open line segment bounded by two successive vertices of that polygon. Furthermore, for purposes of this proof, define an *element* of K to be either a vertex of K or an edge of K. Consider any wedge, with rays PX and PY where X and Y are successive special points along the boundary of H. The wedge consists of the closed convex hull of the two bounding rays (see, for example, the shaded region in figures 6 and 7).

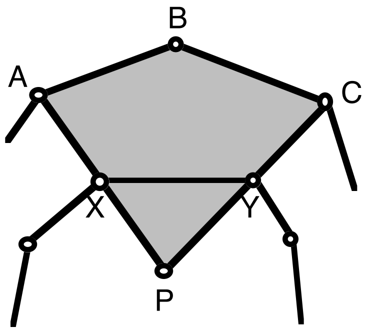

figure 6

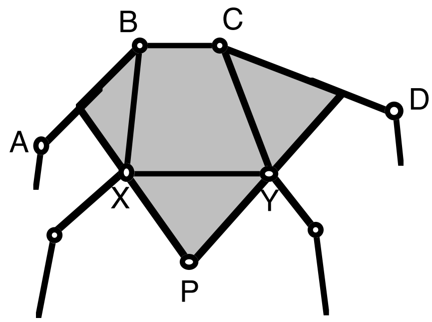

figure 7

Claim. *No wedge intercepts 5 or more elements of K.*

The elements of K alternate: vertex, edge, vertex, ...

Assume that some wedge intercepts 5 or more elements of K. Then these elements either occur in the order (vertex, edge, vertex, edge, vertex, ...) or (edge, vertex, edge, vertex, edge, ...).

First note that a wedge cannot intercept the elements: (vertex, edge, vertex, edge, vertex) since that would contradict the 3-vertex Restriction. See figure 6.

Now we will show that a wedge cannot intercept the elements: (edge, vertex, edge, vertex, edge) for, in that case, there would be two vertices of K, say B and C in the interior of the wedge. See figure 7. Polygon $PXBCY$ is a special pentagon, so by the Quadrangular Segment Theorem, there must be a special point Q inside quadrilateral $BCYX$ or on XY.

If Q were inside quadrilateral $BCYX$, this would be a contradiction, because it would be a special point inside K but outside H. If Q were on XY, this would be a contradiction since X and Y were consecutive special points along the boundary of H. Thus we have arrived at a contradiction in each case.

Hence each wedge intercepts at most 4 elements of K. On the other hand, each ray intercepts exactly one element. Thus the elements intercepted by these rays will be counted twice if we add up all the elements intercepted by the wedges. There are exactly $b(H)$ such elements. Thus the total number of elements in all can't be more than $4b(H)$ minus $b(H)$. But, the total number of elements is just $2v(K)$, so $2v(K) \leq 3b(H)$.

5. The Octagon

Theorem 7. If $v(K) = 8$, then $g(K) \geq 4$.

Proof. Let $ABCDEFGH$ be a special octagon. Then $ABCDEFG$ is a special 7-gon and must contain at least 4 special points in its interior by Theorem 3. Thus K contains at least 4 interior special points.

Theorem 8 (The Octagon Anomaly). A special octagon cannot have exactly 5 interior special points.

The proof of this is the same as the proof given in [10], so we will not repeat it here.

Theorem 9. If a special point lies in the interior of an edge of a special octagon, K, then $g(K) \geq 6$.

Proof. Let $K = ABCDEFGH$ and suppose the special point P lies in the interior of edge AH. By the Quadrangular Segment Theorem applied to pentagon $PEFGH$, there must be a special point, X, inside quadrilateral $PFGH$ or on segment PF. See figure 8. Then $XPABCDE$ is a special heptagon and must contain at least 4 special points in its interior. These plus X show that K contains at least 5 interior special points. But by the Octagon Anomaly, the octagon, K cannot contain exactly 5 special points in its interior; hence $g(K) \geq 6$.

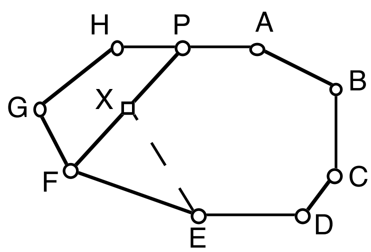

figure 8

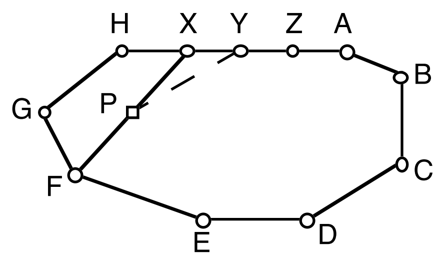

figure 9

Theorem 10. If three special points lie in the interior of an edge of a special octagon, K, then $g(K) \geq 7$.

Proof. Let the octagon be $ABCDEFGH$ and suppose special points X, Y, and Z lie in the interior of edge AH, with the points lying in the order $AZYXH$ as shown in figure 9. Draw FX. By the Quadrangular Segment Theorem applied to pentagon $XEFGH$, there is a special point, P, inside quadrilateral $FGHX$ or on chord XF. In either case, $PYABCDEF$ is a special octagon with a special point on side YA and hence must contain at least 6 interior special points by Theorem 9. Thus, the original octagon contains at least 7 interior special points (these 6 points plus P).

6. The Nonagon

Theorem 11. If $v(K) = 9$, then $g(K) \geq 7$.

Proof. By the Interior Hull Inequalities, we see that $v(H) \geq 5$ and $b(H) \geq 6$. But $v(H) \geq 5$ implies that $g(H) \geq 1$ by the Pentagon Property. Thus $g(K) = b(H) + g(H) \geq 7$.

7. The Decagon

Theorem 12. If $v(K) = 10$, then $g(K) \geq 10$.

Proof. Let H be the interior hull of $K = ABCDEFGHIJ$, a special decagon. Then $b(H) \geq 7$ and $v(H) \geq 5$ by the Interior Hull Inequalities. If $v(H) \geq 7$, then $g(H) \geq 4$, so $g(K) \geq 11$ and we would be done.

Hence we may assume that $v(H)$ is 5 or 6. Since $b(H) \neq v(H)$, we see that H has some edge, say XY, that contains an interior special point Z. By the 3-vertex Restriction, the open halfplane, $h(XY)$, contains at most 2 vertices of K (say A and B).

Suppose XY passes through one vertex of K (say C) or no vertices of K (figure 10). Then $h^*(XY)$ contains at least 7 vertices of K. These 7 vertices plus X and Y yield a special 9-gon which must contain at least 7 interior special points. These plus X, Y, and Z would yield 10 special points in the interior of K.

Suppose XY passes through two vertices of K, say C and J (figure 11). Then $h^*(XY)$ contains at least 6 vertices of K. These plus C and J yield a special octagon $CDEFGHIJ$ with three special points (X, Y, and Z) in the interior of one edge CJ. By Theorem 10, there would be at least 7 special points in the interior of this octagon. These plus X, Y, and Z would yield 10 special points in the interior of K.

In every case, we have found at least 10 special points in the interior of K.

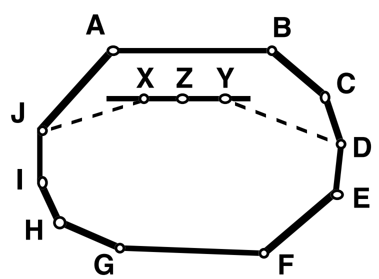

figure 10

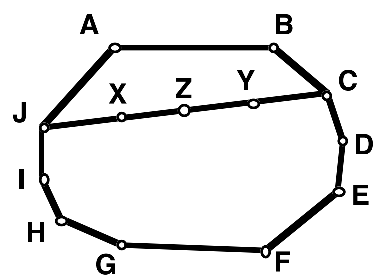

figure 11

8. Larger polygons

A special polygon is said to be *fat* if it contains a special point on the boundary that is not a vertex.

Let $g(n)$ denote the largest integer n such that a special n-gon must necessarily contain at least $g(n)$ special points.

The Fat Hull Lemma. Let H be the interior hull of a special n-gon K. If H is fat, then $g(K) \geq g(n-2) + 4$.

Proof. Let K be a special n-gon. Let H be the interior hull of K. If H is fat, then H has some edge XZ that contains an interior special point, Y.

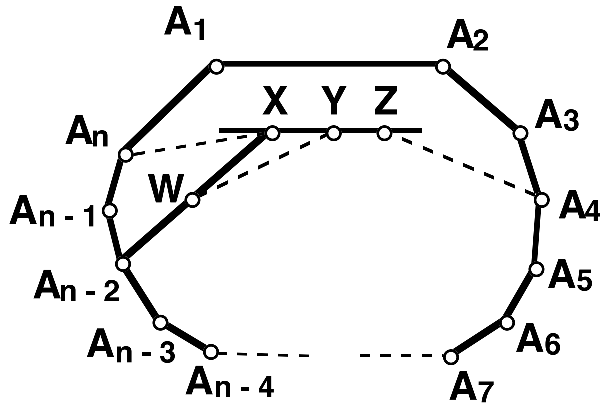

figure 12

By the 3-vertex Restriction, $h(XZ)$ contains at most 2 vertices of K. Label the vertices of K as $A_1A_2A_3A_4...A_n$ such that A_1 and A_2 are the two vertices (at most) that lie in $h(XZ)$. Since XZ passes through at most 2 vertices of K, this means that there are at least $n-4$ vertices of K in $h^*(XZ)$. These vertices are A_4, A_5, $A_6, \ldots,$ A_{n-2}, and A_{n-1}. Line XZ may or may not pass through vertices A_n and A_3. See figure 12.

The points $XA_{n-3}A_{n-2}A_{n-1}A_n$ form a special pentagon. By the Quadrangular Segment Theorem, there is a special point, W, in the interior of quadrilateral $XA_{n-2}A_{n-1}A_n$ or on segment XA_{n-2}. Polygon $WYZA_4A_5A_6 \ldots A_{n-3}A_{n-2}$ is a special $(n-2)$-gon and therefore contains at least $g(n-2)$ special points in its interior. These $g(n-2)$ special points plus W, X, Y, and Z show that K contains at least $g(n-2)+4$ special points in its interior.

Theorem 13. For $n \geq 9$,

$$g(n) \geq \min\bigl(2n/3 + g(\lceil(2n/3\rceil),\ g(n-2)+4\bigr).$$

Proof. Let H be the interior hull of K. By the Interior Hull Inequalities, we have $v(H) \geq n/2$ and $b(H) \geq 2n/3$.

Case 1: $v(H) \geq 2n/3$. Then $v(H) \geq \lceil 2n/3 \rceil$ and

$$g(K) = b(H) + g(H) \geq 2n/3 + g(\lceil 2n/3 \rceil).$$

Case 2: $v(H) < 2n/3$. Then $v(H) < b(H)$, so H is fat. By the Fat Hull Lemma, we have

$$g(K) \geq g(n-2) + 4.$$

Theorem 14. For all n, $g(n) \geq 2n - 12$.

Proof. We proceed by induction on n. We have already proven this result for $n \leq 10$. Assume the result is true for all values less than n and we will now show it is also true for n. By Theorem 13, we have

$$\begin{aligned} g(K) &\geq \min\big(2n/3 + g(\lceil (2n/3 \rceil), 4 + g(n-2)\big) \\ &\geq \min\big(2n/3 + 2\lceil 2n/3 \rceil - 12, 4 + 2(n-2) - 12\big) \\ &\geq \min\big(2n/3 + 2(2n/3) - 12, 2n - 12\big) \\ &\geq \min(2n - 12, 2n - 12) = 2n - 12. \end{aligned}$$

Summary. Using Theorems 13 and 14, we have shown that $g(6) \geq 1$, $g(7) \geq 4$, $g(8) \geq 4$, $g(9) \geq 7$, $g(10) \geq 10$, $g(11) \geq 11$, $g(12) \geq 12$, $g(13) \geq 15$, $g(14) \geq 16$, $g(15) \geq 19$ and $g(16) \geq 20$.

Open Question 1. Can we improve on the bound in Theorem 14?

Open Question 2. For the integer lattice, $\mathbb{Z}^2$, it is known [13] that $g(12) = 19$. However, for $\mathfrak{S}$, the best we have been able to show is that $g(12) \geq 12$. Can we improve on this bound or is this the best we can get using only the Pentagon Property?

Open Question 3. For the integer lattice, $\mathbb{Z}^2$, it is known ([1], [3], [4], [5], [6], [9], and [11]) that there is a constant $c > 0$ such that $g(n) \geq cn^3$. Can we get a similar bound for $\mathfrak{S}$?

Open Question 4. If a countably infinite set of points satisfies the Pentagon Property (with perhaps some additional constraints), must the points in some sense be equivalent to $\mathbb{Z}^2$, say by an integral unimodular affine transformation?

References

[1] G. E. Andrews, "A Lower Bound for the Volumes of Strictly Convex Bodies with Many Boundary Points", *Transactions of the American Mathematical Society*. **106**(1963)270–279.

[2] J. R. Arkinstall, "Minimal Requirements for Minkowski's Theorem in the Plane I", *Bulletin of the Australian Mathematical Society*. **22**(1980)259–274.

[3] V. I. Arnol'd, "Statistics of Integral Convex Polygons", *Functional Analysis & its Applications*. **14**(1980)79–80.

[4] Imre Bárány and János Pach, "On the Number of Convex Lattice Polygons", *Combinatorics, Probability and Computing*. **1**(1992)295–302.

[5] Imre Bárány and Norihide Tokushige, "The Minimum Area of Convex Lattice n-Gons", *Combinatorica*. **24**(2004)171–185.

[6] Tian-Xin Cai, "On the Minimum Area of Convex Lattice Polygons", *Taiwanese Journal of Mathematics*. **1**(1997)351–354.

[7] Donald B. Coleman, "Stretch: A Geoboard Game", *Mathematics Magazine*. **51**(1978)49–54.

[8] Stanley Rabinowitz, "On the Number of Lattice Points Inside a Convex Lattice n-gon", *Congressus Numerantium*. **73**(1990)99–124.

[9] Stanley Rabinowitz, "$O(n^3)$ Bounds for the Area of a Convex Lattice n-gon", *Geombinatorics*. **2**(1993)85–88.

[10] Stanley Rabinowitz, "The Octagon Anomaly", *Geombinatorics*. **3**(1993)13–17.

[11] W. Schmidt, "Integral Points on Surfaces and Curves", *Monatshefte für Mathematik*. **99**(1985)45-72.

[12] P. R. Scott, "On Convex Lattice Polygons", *Bulletin of the Australian Mathematical Society*. **15**(1976)395-399.

[13] R. J. Simpson, "Convex Lattice Polygons of Minimum Area", *Bulletin of the Australian Mathematical Society*. **42**(1990)353–367.