On The Computer Solution of Symmetric Homogeneous Triangle Inequalities

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1. INTRODUCTION

A brief scan through the problem columns of various mathematical journals shows a continual outpouring of proposed problems involving what can be roughly described as "triangle inequalities". A few examples will suffice to indicate the flavor of these perennial favorites:

Let the sides of a triangle be a, b, c and let the angles be A, B, C. Let r and R be the inradius and circumradius, respectively. Let K denote the area of the triangle and let s denote its semiperimeter. Then

(Mavlo [20]) $$(abc)^2 \geq (6rR)^3$$

(Mitrinović [22]) $$\sin^2 2A + \sin^2 2B + \sin^2 2C \geq 36(\frac{r}{R})^4$$

(Krishna [18]) $$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \leq \frac{1}{4r^2}$$

(Milosevic [21])
$$\sin^4 \frac{A}{2} + \sin^4 \frac{B}{2} + \sin^4 \frac{C}{2} \leq \frac{4R^2 - 8Rr + 3r^2}{4R^2}.$$

A perusal through the "bible" of such geometric inequalities, [5], unearths hundreds of such inequalities. It is interesting to note, that the proofs given in [5] for these inequalities are all different and use several different techniques many of which require ingenuity.

It is the purpose of this note to make progress toward coming up with an effective systematic algorithm that one can use to prove all of the above inequalities as well as many others. We present a computer algorithm that, at this time, can prove many (although not all) such inequalities.

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2. BACKGROUND

There has been much investigation into the area of using the computer to mechanically prove theorems in algebra and geometry. Recently, Chou [10] has implemented a computer program that can prove large numbers of advanced geometry theorems. His method involves coordinatization of the problem and then proving an equivalent algebraic formulation. Unfortunately, his algorithm does not handle inequalities.

The types of inequalities we are interested in fall into the study of real closed fields. An inequality such as $x^2+y^2+z^2 \geq xy+yz+zx$ can be formally represented in such a field by the formula

$$(\forall x)(\forall y)(\forall z)\big((x \geq 0) \wedge (y \geq 0) \wedge (z \geq 0) \Rightarrow x^2 + y^2 + z^2 \geq xy + yz + zx\big).$$

The symbols such as $\forall$ and $\exists$ are known as quantifiers. In 1930, Tarski discovered a decision procedure for the formal theory of real closed fields. His method was first published in 1948 [31]. Tarski's method involves successive elimination of quantifiers using Sturm sequences. This method can be used to automatically prove all the inequalities discussed in this paper as well as many problems in Euclidean geometry. Unfortunately, Tarski's algorithm, although totally effective, is completely impractical given the state of the art of computers today. The same holds true for an improvement given by Seidenberg, [30].

Over the years, improved methods (such as using Gröbner bases) have been devised for effectively proving results in the theory of real closed fields. Perhaps the best method is one devised by Collins [12] which employs cylindrical decomposition to eliminate quantifiers. See Davenport [13], section 3.2, for an exposition. Consult [11] for a guide to the literature in this area.

Even the method of cylindrical decomposition may be too slow for proving the types of inequalities we are interested in. That is because our triangle inequalities generally involve three quantifiers, and these methods attempt to eliminate one quantifier at a time. Each elimination step causes an expression explosion. In our case, the expressions involved are symmetric; so a method that removes one quantifier at a time is bound to be non-optimal. We will attempt below (in section 5) to exploit the symmetry of the problem in proving these inequalities.

Blundon and others ([4], [6], [27]) have attempted to devise an effective algorithm by first expressing the proposed inequality in terms of R, r, and s. It involves Blundon's fundamental inequality:

$$s^2(18Rr - 9r^2 - s^2)^2 \leq (s^2 - 12Rr - 3r^2)^3.$$

See chapter 1 of [23] for more details.

Another important approach to solving triangle inequalities is through the method of majorization. The algorithm described in this paper uses majorization only in the context of Muirhead's Theorem. Many other majorizations apply and can be used to prove many triangle inequalities quite quickly. The interested reader should consult [19]. However, I do not see yet how these more general algorithms can be used to produce an effective computer solution.

3. NOTATION

First we shall specify the class of inequalities that this algorithm will handle.

Definition. A function, $f(x_1, x_2, x_3, \ldots, x_n)$, of n variables is *homogeneous of degree k* if

$$f(tx_1, tx_2, \ldots, tx_n) \equiv t^k f(x_1, x_2, \ldots, x_n).$$

Clearly, a polynomial in n variables is homogeneous of degree k if and only if each term has degree k. (The degree of a term is the sum of the exponents of the variables appearing in that term.)

Definition. A function of n variables is *symmetric* if it's value remains unchanged when any permutation is applied to the variables.

For example, a function of 3 variables, $f(x, y, z)$, is symmetric if

$$\begin{aligned} f(x,y,z) &= f(x,z,y) = f(y,x,z) \\ &= f(y,z,x) = f(z,x,y) = f(z,y,x). \end{aligned}$$

If a, b, and c are the sides of a triangle, then the class of functions that we will be concerned with are those functions that can be expressed as homogeneous symmetric polynomials in a, b, and c.

If we do not restrict ourselves to homogeneous polynomials, then we are essentially dealing with the general case of arbitrary polynomial inequalities. For example, the inequality

$$x^2 + y^2 + z^2 + 3 \geq 2(x + y + z)$$

is equivalent to the univariate polynomial inequality

$$x^2 + 1 \geq 2x.$$

In general, homogeneity alone may not simplify the general problem. For example, suppose we come up with an algorithm that always works for symmetric (not necessarily homogeneous) inequalities. Then it should also work when there are just two variables, say x and y. Then we would have an algorithm for proving a symmetric polynomial inequality of the form $f(x, y) > 0$. But letting $y = tx$, we would find that this leads us to a proof of a polynomial inequality in one variable, t. Conversely, starting with any univariate polynomial inequality, $F(t) > 0$, we may let $t = x/y$, multiply through by an appropriate power of y and get an equivalent symmetric polynomial inequality, $f(x, y) > 0$.

The inequalities that we will be able to verify are of the form

$$f(a, b, c) \succeq 0$$

where the function f is a homogeneous symmetric polynomial in a, b, and c and the symbol "$\succeq$" can be any of the arithmetic relations "$>$", "$\geq$", "$<$", "$\leq$", or "$=$".

Convention. An inequality expressed with the variables x_i or with the variables x, y, and z shall be understood to represent an inequality that is valid whenever the variables are non-negative. In other words, it is always to be understood that $x_i \geq 0$ and $x \geq 0$, $y \geq 0$, $z \geq 0$. An inequality expressed with the variables a, b, and c shall be understood to represent an inequality that is valid whenever the variables are non-negative and satisfy the triangle inequality. In other words, it is always to be understood that $a, b, c \geq 0$, $a + b \geq c$, $b + c \geq a$, and $c + a \geq b$.

Definition. If

$$\prod_{i=1}^{n}(t + x_i) = \sum_{i=0}^{n} \tau_i t^{n-i}$$

then $\tau_r = \tau_r(x_1, x_2, \ldots, x_n)$ is called the rth elementary symmetric function of the x_i. In other words, τ_r is the sum of the products, taken r at a time, of the x_i.

Notation. Following Hardy, Littlewood, and Polya [15], we shall use the notation

$$\sum ! \, f(x_1, x_2, \ldots, x_n)$$

to denote the symmetric sum obtained from the specified expression by adding together the $n!$ expressions obtained from the specified one by applying all the permutations of $(x_1, x_2, \ldots, x_n)$.

The standard sigma notation,

$$\sum f(x_1, x_2, \ldots, x_n),$$

with no expressions above or below the sigma symbol, shall represent the cyclic sum of the specified expression. This is obtained by adding together the expressions obtained by successive applications of the cyclic permutation $(x_1, x_2, \ldots, x_n) \to (x_2, x_3, \ldots, x_n, x_1)$.

Thus, for example, in the case of 3 variables, x, y, and z, we have

$$\sum x^2 y \equiv x^2 y + y^2 z + z^2 x$$

and

$$\sum ! \, x^2 y \equiv x^2 y + y^2 x + y^2 z + z^2 y + z^2 x + x^2 z.$$

If

$$f(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} x_i^{r_i}$$

is a product of variables, then

$$\sum! f(x_1, x_2, \ldots, x_n)$$

is known as a *simple symmetric sum*. A simple symmetric sum may also be referred to as a *symmetric term*.

Note that a simple symmetric sum of n variables always contains $n!$ terms (counting multiplicities). In particular, when $n = 3$, the homogeneous symmetric sums involving the elementary symmetric polynomials are:

$$\sum! x = 2(x+y+z),$$

$$\sum! xy = 2(xy+yz+zx),$$

and

$$\sum! xyz = 6xyz.$$

Notation. Let $[r_1, r_2, \ldots, r_n]$ denote a vector of real numbers with the property that $r_1 \geq r_2 \geq \cdots \geq r_n \geq 0$. Such a vector is said to be in *decreasing form*.

Note that the symmetric terms $\sum! x_1^{q_1} x_2^{q_2} \cdots x_n^{q_n}$ and $\sum! x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n}$ are identical if $(r_1, r_2, \ldots, r_n)$ is a permutation of $(q_1, q_2, \ldots, q_n)$. We can therefore pick a canonical permutation, namely the one in which the exponents are non-increasing. We shall always do this in the future. In this manner, we can identify a simple symmetric sum (of a product of n variables) with a decreasing n-tuple as follows:

$$[r_1, r_2, \ldots, r_n] \equiv \sum! x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n}$$

where $r_1 \geq r_2 \geq \cdots \geq r_n \geq 0$.

Sometimes in the literature, a factor of $\frac{1}{n!}$ is included with the summation sign. Since all our inequalities are homogeneous, we will omit such a numerical factor from our definition. This will have no affect on the resulting inequality.

We note some properties of simple symmetric sums. Properties 1 and 2 are presented for the case $n = 3$ but are clearly true for arbitrary n.

Property 1. If d is a real number and

$$\sum_{i=1}^{m} c_i [p_i, q_i, r_i] \geq 0$$

then

$$\sum_{i=1}^{m} c_i [p_i + d, q_i + d, r_i + d] \geq 0.$$

Proof. This follows by multiplying both sides of the inequality by the positive term $x^d y^d z^d$. □

Property 2. If

$$\sum_{i=1}^{m} c_i [p_i, q_i, r_i] \geq 0$$

then

$$\sum_{i=1}^{m} c_i [-p_i, -q_i, -r_i] \geq 0.$$

Proof. This follows by letting $x = 1/X$, $y = 1/Y$, and $z = 1/Z$. □

Property 3. If we have $[p'_1, \ldots, p'_j] \leq [p_1, \ldots, p_j]$ and $[q'_1, \ldots, q'_k] \leq [q_1, \ldots, q_k]$ then $[p'_1, \ldots, p'_j, q'_1, \ldots, q'_k] \leq [p_1, \ldots, p_j, q_1, \ldots, q_k]$.

For a proof, see Hardy, Littlewood and Pólya [15], page 63.

4. MUIRHEAD'S THEOREM

Definition. Let the vectors $p = [p_1, p_2, \ldots, p_n]$ and $q = [q_1, q_2, \ldots, q_n]$ be two decreasing vectors of length n. We say that p is *majorized* by q (or that q *majorizes* p) and write $p \prec q$ (or $q \succ p$) if

(i) $p_1 + p_2 + \cdots + p_n = q_1 + q_2 + \cdots + q_n$ and

(ii) $p_1 + p_2 + \cdots + p_k \leq q_1 + q_2 + \cdots + q_k \quad (1 \leq k < n)$.

Muirhead's Theorem. *Let $p = [p_1, p_2, \ldots, p_n]$ and $q = [q_1, q_2, \ldots, q_n]$ be two decreasing vectors of real numbers identified with their corresponding simple symmetric sums. Then $p \leq q$ for all positive values of the x_i if and only if $p \prec q$. Equality holds when and only when either $p = q$ or all the x_i are equal.*

For a proof, see Hardy, Littlewood and Pólya [15], pp. 46–48 or [9], section 5.7. Muirhead [26] first proved this theorem in 1903 for the case where all the exponents are positive integers. See also [33] for a simple proof of this case. Inequalities that are a direct consequence of Muirhead's Theorem will be referred to as *Muirhead inequalities*.

Example.

Prove: $x^2 + y^2 + z^2 \geq xy + yz + zx$.

Solution: $\sum x^2 = \frac{1}{2}\sum! x^2 = \frac{1}{2}[2,0,0]$. $\sum xy = \frac{1}{2}\sum! xy = \frac{1}{2}[1,1,0]$. But $[2,0,0] \succ [1,1,0]$, so $\frac{1}{2}[2,0,0] \geq \frac{1}{2}[1,1,0]$.

5. A COMPUTER ALGORITHM

In this section, we describe an algorithm (more technically a set of heuristics) for proving certain polynomial inequalities. We will also discuss cases where this "algorithm" fails.

Algorithm P (for Symmetric Homogeneous Polynomial Inequalities).

We are given a polynomial $f(x_1, x_2, \ldots, x_n)$ in n variables that is symmetric and homogeneous, for some fixed constant n. We wish to prove or disprove the inequality $f(x_1, x_2, \ldots, x_n) \geq 0$ for all positive values of the x_i.

Step 1: Expand out the polynomial as a sum of terms and collect together all formally positive terms on one side of the inequality and all the other terms on the other side. We are left with an inequality of the form $f(x_1, x_2, \ldots, x_n) \geq g(x_1, x_2, \ldots, x_n)$ where f and g are symmetric homogeneous polynomials.

Step 2: Express each side explicitly as a symmetric polynomial. Replace each symmetric term in each symmetric polynomial by its associated vector. If one side of the inequality is now 0 (and this is the side to the right of the "$\geq$" operator), then the inequality is declared to be true because a sum of positive terms must be positive. Otherwise, proceed to step 3.

Step 3: Use Muirhead's Theorem to decide the truth or falsity of the resulting inequality.

Unfortunately, step 3 is not effective. We shall see why in a moment.

Example 1.
Prove: $(x+y)(y+z)(z+x) \geq 8xyz$.
Solution by Algorithm P: Expanding out the left side and then bringing all xyz terms to the right gives

$$x^2y + xy^2 + y^2z + yz^2 + z^2x + zx^2 \geq 6xyz.$$

Expressing this using our vector notation for simple symmetric sums, we get $[2,1,0] \geq [1,1,1]$. This is now immediately seen to be true by Muirhead's Theorem because $[2,1,0] \succ [1,1,1]$.

Example 2.
Prove: $(x+y+z)^2 \geq x^2+y^2+z^2$.
Solution by Algorithm P: When we expand and collect like terms, we find that we are left with $2xy+2yz+2zx \geq 0$. Since the right side is 0 and the left side contains only non-negative terms, we can stop at this point and declare the inequality true.

Example 3.
Prove: $(x+y+z)^3 \geq 27xyz$.
Solution by Algorithm P: Expanding out and collecting like terms gives:

$$x^3+y^3+z^3+3x^2y+3xy^2+3y^2z+ \\ 3yz^2+3z^2x+3zx^2 \geq 21xyz.$$

Expressing in terms of simple symmetric sums yields:

$$[3,0,0]+6[2,1,0] \geq 7[1,1,1]$$

which we note follows immediately by adding together two inequalities that come from Muirhead's Theorem, namely

$$[3,0,0] \geq [1,1,1]$$

$$6[2,1,0] \geq 6[1,1,1].$$

Example 4.
Prove: $xyz \geq (x+y-z)(y+z-x)(z+x-y)$.
Attempted solution by Algorithm P: Expanding out and then collecting like terms together gives:

$$x^3+y^3+z^3+3xyz \geq x^2y+xy^2+y^2z+yz^2+z^2x+zx^2.$$

When expressed as simple symmetric sums, we have

$$[3,0,0]+[1,1,1] \geq 2[2,1,0].$$

At this point we are stuck, since this inequality does not follow from Muirhead's Theorem nor does it follow from a sum of inequalities each determined by Muirhead's Theorem.

Thus we see why step 3 of Algorithm P is not totally effective. The reason is that we get an equivalent inequality each side of which is a sum of symmetric terms. Muirhead's theorem only applies when there is one symmetric term on each side. In other words, the validity of $a+b \geq c+d$ might follow because $a \succ c$ and $b \succ d$ or it might follow because $a \succ d$ and $b \succ c$ or it might follow for some other reason that has nothing to do with Muirhead's Theorem. Thus my algorithm is not effective.

It would be nice if one could determine a necessary and sufficient condition for when a sum of symmetric terms was larger than another sum of symmetric terms. Unfortunately, no such necessary and sufficient condition is known. One sufficient condition that occurs frequently is known as Schur's Inequality.

Schur's Inequality. If p and d are positive real numbers, then

$$[p+2d,0,0]+[p,d,d] \geq 2[p+d,d,0].$$

Proof. Expanding out, we see that we wish to prove

$$2\sum x^{p+2d} + 2\sum x^p y^d z^d \geq 2\sum x^{p+d}(y^d+z^d).$$

This is equivalent to

$$\sum x^p(x^{2d}+y^dz^d-x^dy^d-x^dz^d) \geq 0.$$

Letting $X=x^d$, $Y=y^d$, $Z=z^d$, we find that this is equivalent to

$$\sum x^\lambda(x-y)(x-z) \geq 0$$

where $\lambda = p/d$ is positive.

Since this expression is symmetric in x, y, and z, we may assume without loss of generality that $x \geq y \geq z$. The third term on the left is $z^\lambda(z-x)(z-y)$ which is clearly non-negative. Since $x^\lambda \geq y^\lambda$, we have for the sum of the first two terms:

$$\begin{aligned} x^\lambda(x-y)(x-z) &+ y^\lambda(y-x)(y-z) \\ &\geq y^\lambda(x-y)(x-z)+y^\lambda(x-y)(z-y) \\ &\geq y^\lambda(x-y)[(x-z)+(z-y)] \\ &= y^\lambda(x-y)^2 \geq 0. \end{aligned}$$

□

The above proof is based on the one in Barnard and Child [3]. For another proof, see Mitrinović [24], section 2.17.

Theorem. If p and d are positive real numbers, then

$$[p+d,p+d,0]+[2p,d,d] \geq 2[p+d,d,p].$$

(We will also refer to this as Schur's Inequality.)

Proof. Start with Schur's inequality:

$$[p+2d,0,0]+[p,d,d]\geq 2[p+d,d,0].$$

By property 2 of simple symmetric sums, this is equivalent to:

$$[-p-2d,0,0]+[-p,-d,-d]\geq 2[-p-d,-d,0].$$

Now use property 1, adding $p+2d$ to each entry gives:

$$[0,p+2d,p+2d]+[2d,p+d,p+d]\geq 2[d,p+d,p+2d].$$

Now let $x=d$, $y=p+d$ to get

$$[0,x+y,x+y]+[2x,y,y]\geq 2[x,y,x+y]$$

which is equivalent to what we wanted to prove. □

Most symmetric homogeneous inequalities seem to follow from a combination of Muirhead inequalities and Schur inequalities. Therefore, I coded step 3 of Algorithm P to look for both types of inequalities. For example, in example 4 above, Algorithm P would note that the inequality is true because of Schur's Inequality.

Sometimes an inequality requires application of both Schur's Inequality and several Muirhead inequalities.

Example 5.
Prove: $3(x^2y+y^2z+z^2x)(xy^2+yz^2+zx^2)\geq xyz(x+y+z)^3$.
Solution by Algorithm P: After expanding out and collecting like terms, we express the result in terms of simple symmetric sums to get

$$2[4,1,1]+3[3,3,0]+[2,2,2]\geq 6[3,2,1].$$

This follows by adding up the following three inequalities:

$$[3,3,0]+[2,2,2]\geq 2[3,2,1]$$
$$2[4,1,1]\geq 2[3,2,1]$$
$$2[3,3,0]\geq 2[3,2,1]$$

the first of which follows from Schur's Inequality and the last two of which follow from Muirhead's Theorem.

We should also note at this time that we might get a large number of symmetric terms on each side. Even if the inequality is true purely by combining a sequence of Schur inequalities and Muirhead inequalities, it might be non-trivial to determine how to do this. I do this via a set of heuristics. Algorithm P goes through the following sequence of heuristics (in order). Each time that it finds one that applies, it subtracts the result from the inequality to be proved and then begins again with the first heuristic. The algorithm terminates when one side of the inequality has gone to 0. If this is the side to the right of the "≥" sign, then the inequality is true. If it is the other side, then the truth or falsity of the proposed inequality has not been determined. The algorithm fails completely if none of the heuristics apply.

The inequality is assumed to start out in the form

$$\sum_{i=1}^{m} c_i t_i \geq \sum_{i=1}^{m'} c'_i t'_i$$

where each t_i and t'_i denotes a symmetric term and the c_i and c'_i are positive numeric constants.

Heuristic 1: See if there is some term, t_i, on the left that does not majorize any term on the right. If that is the case, then see if t_i can be paired with some t_j on the left, so that $t_i+t_j\geq 2t'_k$ is true by Schur's Inequality where t'_k occurs on the right. If so, subtract off c times this Schur inequality, $t_i+t_j\geq 2t_k$, where $c=\min(c_i,c_j,\frac{1}{2}c_k)$.

Heuristic 2: If there is only one term on the right, $c't'$, then for each term, t_i, on the left that majorizes t', subtract off the inequality $ct_i\geq ct'$ where $c=\min(c',c_i)$. Continue doing this until the right-hand side goes to 0.

Heuristic 3: If there is a term on the right, t', that is majorized by precisely one term, t_i, on the left, then subtract off the inequality $ct_i\geq ct'$ where $c=\min(c',c_i)$.

Heuristic 4: If there is a term on the left, t_i, that majorizes precisely one term, t'_j, on the right, then subtract off the inequality $ct_i\geq ct'_j$ where $c=\min(c_i,c'_j)$.

Heuristic 5: If every term on the left majorizes every term on the right, then subtract off the inequality $ct_1\geq ct'_1$ where $c=\min(c_1,c'_1)$.

Heuristic 6: If some term, t_i, on the left majorizes some term, t'_j, on the right then subtract off the inequality $ct_i\geq ct'_j$ where $c=\min(c_i,c'_j)$.

6. EXAMPLES

To test Algorithm P, I coded it up using Macsyma (on an Alliant FX/80) and fed it the following 27 examples obtained from Hall and Knight [14] and Barnard and Child [3]. For each one, I list the resulting inequality that was obtained and why it follows from one or more applications of Muirhead's Theorem or Schur's Inequality. Algorithm P automatically proved 26 out of the 27 examples.

Theorem P.

$$x^2+y^2\geq 2xy \quad (1)$$
$$x^3y+xy^3\leq x^4+y^4 \quad (2)$$
$$(x+y)(y+z)(z+x)\geq 8xyz \quad (3)$$
$$4(x^3+y^3)\geq (x+y)^3 \quad (4)$$
$$(x+y+z)^3\geq 27xyz \quad (5)$$
$$xyz\geq (x+y-z)(y+z-x)(z+x-y) \quad (6)$$
$$(x^3+y^3)^2\leq (x^2+y^2)^3 \quad (7)$$
$$8(x^3+y^3+z^3)^2\geq 9(x^2+yz)(y^2+zx)(z^2+xy) \quad (8)$$
$$4(x^5+y^5+z^5+w^5)\geq (x^3+y^3+z^3+w^3)(x^2+y^2+z^2+w^2) \quad (9)$$

$$(y+z+w)(z+w+x)(w+x+y)(x+y+z) \geq 81wxyz \quad (10)$$

$$(x^2y + y^2z + z^2x)(xy^2 + yz^2 + zx^2) \geq 9x^2y^2z^2 \quad (11)$$

$$6xyz \leq xy(x+y) + yz(y+z) + zx(z+x) \quad (12)$$

$$x^2y^2 + y^2z^2 + z^2x^2 \geq xyz(x+y+z) \quad (13)$$

$$(x+y+z)^3 \geq (x+y-z)(y+z-x)(z+x-y) \quad (14)$$

$$27(x^4+y^4+z^4) \geq (x+y+z)^4 \quad (15)$$

$$(x+y+z+w)(x^3+y^3+z^3+w^3) \geq (x^2+y^2+z^2+w^2)^2 \quad (16)$$

$$x(x-y)(x-z)+y(y-z)(y-x)+z(z-x)(z-y) \geq 0 \quad (17)$$

$$x^2(x-y)(x-z)+y^2(y-z)(y-x)+z^2(z-x)(z-y) \geq 0 \quad (18)$$

$$x^5(x-y)(x-z)+y^5(y-z)(y-x)+z^5(z-x)(z-y) \geq 0 \quad (19)$$

$$9(x^6+y^6+z^6) \geq (x^3+y^3+z^3)(x^2+y^2+z^2)(x+y+z) \quad (20)$$

$$16(x^3+y^3+z^3+w^3) \geq (x+y+z+w)^3 \quad (21)$$

$$(x+y-z)^2+(y+z-x)^2+(z+x-y)^2 \geq xy+yz+zx \quad (22)$$

$$x^3+y^3+z^3+3xyz \geq x^2(y+z)+z^2(x+y)+y^2(z+x) \quad (23)$$

$$6xyz \leq x^2(y+z)+y^2(z+x)+z^2(x+y) \quad (24)$$

$$x^2(y+z)+y^2(z+x)+z^2(x+y) \leq 2(x^3+y^3+z^3) \quad (25)$$

$$3(x^2y+y^2z+z^2x)(xy^2+yz^2+zx^2) \geq xyz(x+y+z)^3 \quad (26)$$

$$(x^7+y^7)^3 \leq (x^3+y^3)^7 \quad (27)$$

Computer Proof.
(1) is equivalent to $[2,0] \geq [1,1]$ which follows from Muirhead's Theorem.
(2) is equivalent to $[4,0] \geq [3,1]$ which follows from Muirhead's Theorem.
(3) is equivalent to $[2,1,0] \geq [1,1,1]$ which follows from Muirhead's Theorem.
(4) is equivalent to $[2,0] \geq [1,1]$ which follows from Muirhead's Theorem.
(5) is equivalent to $[3,0,0] + 6[2,1,0] \geq 7[1,1,1]$ which follows from the majorizations

$$[3,0,0] \succ [1,1,1].$$

$$6[2,1,0] \succ 6[1,1,1]$$

(6) is equivalent to $[3,0,0] + [1,1,1] \geq 2[2,1,0]$ which follows from Schur's Inequality.
(7) is equivalent to $3[2,0] \geq [1,1]$ which follows from Muirhead's Theorem.
(8) is equivalent to $8[6,0,0] + 7[3,3,0] \geq 9[4,1,1] + 6[2,2,2]$ which algorithm P cannot prove yet automatically. The truth of this inequality follows from the inequalities:

$$[6,0,0] + [2,2,2] \geq 2[4,2,0]$$

$$2[4,2,0] \geq 2[4,1,1]$$

$$7[6,0,0] \geq 7[4,1,1]$$

$$7[3,3,0] \geq 7[2,2,2].$$

(9) is equivalent to $[5,0,0,0] \geq [3,2,0,0]$ which follows from Muirhead's Theorem.
(10) is equivalent to $[3,1,0,0]+[2,2,0,0]+4[2,1,1,0] \geq 6[1,1,1,1]$ which follows from the majorizations

$$[3,1,0,0] \succ [1,1,1,1].$$

$$[2,2,0,0] \succ [1,1,1,1]$$

$$4[2,1,1,0] \succ 4[1,1,1,1]$$

(11) is equivalent to $[4,1,1] + [3,3,0] \geq 2[2,2,2]$ which follows from the majorizations

$$[4,1,1] \succ [2,2,2]$$

$$[3,3,0] \succ [2,2,2].$$

(12) is equivalent to $[2,1,0] \geq [1,1,1]$ which follows from Muirhead's Theorem.
(13) is equivalent to $[2,2,0] \geq [2,1,1]$ which follows from Muirhead's Theorem.
(14) is equivalent to $2\sum x^3 + 2\sum! \, x^2y + 8xyz \geq 0$ which follows because a sum of positive terms is positive.
(15) is equivalent to $13[4,0,0] \geq 4[3,1,0] + 3[2,2,0] + 6[2,1,1]$ which follows from the majorizations

$$4[4,0,0] \succ 4[3,1,0]$$

$$3[4,0,0] \succ 3[2,2,0]$$

$$6[4,0,0] \succ 6[2,1,1].$$

(16) is equivalent to $[3,1,0,0] \geq [2,2,0,0]$ which follows from Muirhead's Theorem.
(17) is equivalent to $[3,0,0] + [1,1,1] \geq 2[2,1,0]$ which follows from Schur's Inequality.
(18) is equivalent to $[4,0,0] + [2,1,1] \geq 2[3,1,0]$ which follows from Schur's Inequality.
(19) is equivalent to $[7,0,0] + [5,1,1] \geq 2[6,1,0]$ which follows from Schur's Inequality.
(20) is equivalent to $4[6,0,0] \geq [5,1,0] + [4,2,0] + [3,3,0] + [3,2,1]$ which follows from the majorizations

$$[6,0,0] \succ [5,1,0]$$

$$[6,0,0] \succ [4,2,0]$$

$$[6,0,0] \succ [3,3,0]$$

$$[6,0,0] \succ [3,2,1].$$

(21) is equivalent to $5[3,0,0,0] \geq 3[2,1,0,0]+2[1,1,1,0]$ which follows from the majorizations

$$3[3,0,0,0] \succ 3[2,1,0,0]$$

$$2[3,0,0,0] \succ 2[1,1,1,0].$$

(22) is equivalent to $[2,0,0] \geq [1,1,0]$ which follows from Muirhead's Theorem.

(23) is equivalent to $[3,0,0]+[1,1,1] \geq 2[2,1,0]$ which follows from Schur's Inequality.
(24) is equivalent to $[2,1,0] \geq [1,1,1]$ which follows from Muirhead's Theorem.
(25) is equivalent to $[3,0,0] \geq [2,1,0]$ which follows from Muirhead's Theorem.
(26) is equivalent to $2[4,1,1] + 3[3,3,0] + [2,2,2] \geq 6[3,2,1]$ which follows from the inequalities

$$[3,3,0]+[2,2,2] \geq 2[3,2,1]$$

$$2[4,1,1] \geq 2[3,2,1]$$

$$2[3,3,0] \geq 2[3,2,1].$$

(27) is equivalent to $7[14,0] + 7[12,2] + 14[11,3] + 17[9,5] + 18[8,6] \geq 7[13,1] + 17[10,4] + 9[7,7]$ which follows from the majorizations

$$7[14,0] \succ 7[13,1]$$

$$7[12,2] \succ 7[10,4]$$

$$10[11,3] \succ 10[10,4]$$

$$4[11,3] \succ 4[7,7]$$

$$5[9,5] \succ 5[7,7].$$

Unfortunately, not all symmetric homogeneous inequalities follow from Muirhead's Theorem and Schur's Inequality alone. An example that fails is

$$x^4+y^4+z^4+3x^2y^2+3y^2z^2+3z^2x^2 \geq 2x^3(y+z)+2y^3(z+x)+2z^3(x+y)$$

which can be expressed as

$$[4,0,0]+3[2,2,0] \geq 4[3,1,0].$$

This inequality is true because it is equivalent to

$$(x-y)^4+(y-z)^4+(z-x)^4 \geq 0.$$

We summarize below additional work that has been done on inequalities consisting of a sum of symmetric terms. The following two theorems come from the work of Albada [1], Rigby [28], and Bottema and Groenman [7]. I have translated their theorems into my notation.

Theorem SH2.
(Characterization of Symmetric Homogeneous Positive Inequalities of degree 2).
A necessary and sufficient condition for

$$a[2,0,0]+b[1,1,0] \geq 0$$

is that

$$a \geq 0$$

$$a+b \geq 0.$$

Proof. The inequality is equivalent to

$$a(x^2+y^2+z^2)+b(xy+yz+zx) \geq 0.$$

Necessity:
$x=1,\ y=z=0 \Rightarrow a \geq 0.$
$x=y=z=1 \Rightarrow a+b \geq 0.$

Sufficiency:

$$a[2,0,0]+b[1,1,0] = a([2,0,0]-[1,1,0])+(a+b)[1,1,0]$$

which is non-negative by Muirhead's Theorem. □

Theorem SH3.
(Characterization of Symmetric Homogeneous Positive Inequalities of degree 3).
A necessary and sufficient condition for

$$a[3,0,0]+b[2,1,0]+c[1,1,1] \geq 0$$

is that

$$a \geq 0$$

$$2a+b \geq 0$$

$$a+b+c \geq 0.$$

Proof. The inequality is equivalent to

$$b(x^2y+xy^2+y^2z+yz^2+z^2x+zx^2)+2a(x^3+y^3+z^3)+6cxyz \geq 0.$$

Necessity:
$x=1,\ y=z=0 \Rightarrow a \geq 0.$
$x=0,\ y=z=1 \Rightarrow 2a+b \geq 0.$
$x=y=z=1 \Rightarrow a+b+c \geq 0.$

Sufficiency:

$$a[3,0,0]+b[2,1,0]+c[1,1,1] = a([3,0,0]-2[2,1,0]+[1,1,1]) +(2a+b)([2,1,0]-[1,1,1]) +(a+b+c)[1,1,1]$$

which is non-negative by Schur's Inequality and Muirhead's Theorem. □

For similar results of higher degree, consult chapter 3 of [23].

Instead of expanding everything in terms of simple symmetric sums, it might be preferable to expand in terms of the elementary symmetric polynomials.

Proposition. Any symmetric polynomial in the variables $x_1, x_2, \ldots, x_n$ can be written uniquely as a polynomial in the elementary symmetric polynomials $\tau_i(x_1, x_2, \ldots, x_n)$.

This was first proven by Gauss. His constructive proof can be found in Mostowski and Stark, [25], page 354. A code fragment, employing Waring's method, can be found in [13], page 14.

Proposition. Let $T_i = \tau_i/\binom{n}{i}$ where τ_i denotes the ith elementary symmetric polynomial in $x_1, x_2, \ldots, x_n$. Then a necessary and sufficient condition that

$$\prod_{i=1}^{n} T_i^{p_i} \leq \prod_{i=1}^{n} T_i^{q_i}$$

when $p_i, q_i \geq 0$ is that

$$p_m + 2p_{m+1} + \cdots (n-m+1)p_n \geq$$
$$q_m + 2q_{m+1} + \cdots + (n-m+1)q_n$$

for $1 \leq m \leq n$ with equality when $m = 1$.

For a proof, see Hardy, Littlewood and Pólya [15], page 64.

Open Question 1. Would it be better to use a method that was based on the representation involving elementary symmetric polynomials instead of one based on symmetric terms?

Algorithm R (for Symmetric Homogeneous Rational Functions).

If each side of the proposed inequality is a rational function (quotient of two polynomials), then we can proceed as follows. If one side of the inequality has the form f/g where f and g are two polynomials, then g is a symmetric homogeneous polynomial. We can apply algorithm P to determine if $g \geq 0$ or not. If $g \geq 0$ is true, then multiply both sides of the inequality by g. If $g \leq 0$ is true, then multiply both sides of the inequality by $(-g)$. If neither $g \geq 0$ nor $g \leq 0$ is true (i.e. if g is not comparable to 0), then we are stymied and the algorithm fails. I have yet to find an example of this kind.

We then do the same thing for the right-hand side of the inequality. We are left with a polynomial inequality and we can now apply algorithm P.

Example.
Prove: $1/x + 1/y + 1/z \geq 9/(x+y+z)$.
Computer solution: Rationally simplify the left-hand side to get $\frac{yz+zx+xy}{xyz}$. The denominator, xyz, is determined to satisfy $xyz \geq 0$ by algorithm P. The denominator of the right-hand side is $x+y+z$ and algorithm P easily shows that $x+y+z \geq 0$. We may thus multiply both sides of the inequality by $xyz(x+y+z)$ to get the polynomial inequality

$$\sum! \; x^2y + 3xyz \geq 9xyz.$$

We now apply algorithm P. Combining like terms gives

$$\sum! \; x^2y \geq 6xyz$$

and this is equivalent to

$$[2,1,0] \geq [1,1,1]$$

which follows from Muirhead's Theorem.

I tested algorithm R on the following 6 inequalities from Hall and Knight [14]. Algorithm R automatically proved all 6 of the inequalities.

Theorem R.

$$\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} \geq \frac{3}{2} \qquad (1)$$

$$\frac{(x-y)(x-z)}{x^5} + \frac{(y-z)(y-x)}{y^5} + \frac{(z-x)(z-y)}{z^5} \geq 0 \qquad (2)$$

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \leq \frac{x^8+y^8+z^8}{x^3y^3z^3} \qquad (3)$$

$$\frac{1}{x+y+z} + \frac{1}{y+z+w} + \frac{1}{z+w+x} + \frac{1}{w+x+y} \geq \frac{16/3}{x+y+z+w} \qquad (4)$$

$$\frac{x^2+y^2}{x+y} + \frac{y^2+z^2}{y+z} + \frac{z^2+x^2}{z+x} \geq x+y+z \qquad (5)$$

$$(x^3+y^3+z^3+v^3+w^3)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{v}+\frac{1}{w}\right) \geq 5(x^2+y^2+z^2+v^2+w^2) \qquad (6)$$

Computer Proof.
(1) is equivalent to $[3,0,0] \geq [2,1,0]$ which follows from Muirhead's Theorem.
(2) is equivalent to $[6,6,0] + [5,5,2] \geq 2[6,5,1]$ which follows from Schur's inequality.
(3) is equivalent to $[8,0,0] \geq [3,3,2]$ which follows from Muirhead's Theorem.
(4) is equivalent to $[4,0,0,0] + 2[3,1,0,0] \geq [2,2,0,0] + [2,1,1,0] + [1,1,1,1]$ which follows from the majorizations

$$[4,0,0,0] \succ [2,2,0,0]$$
$$[3,1,0,0] \succ [2,1,1,0]$$
$$[3,1,0,0] \succ [1,1,1,1].$$

(5) is equivalent to $[3,1,0] \geq [2,1,1]$ which follows from Muirhead's Theorem.
(6) is equivalent to $[4,1,1,1,0] \geq [3,1,1,1,1]$ which follows from Muirhead's Theorem.

It should be noted that there is a heuristic that speeds up Algorithm R a slight amount. If the denominators of the left and right-hand sides are D_1 and D_2, then instead of multiplying both sides of the inequality by D_1D_2, it suffices to multiply both sides by $\mathrm{lcm}(D_1, D_2)$. This least common multiple can be found by the formula $\mathrm{lcm}(D_1, D_2) = D_1D_2/\gcd(D_1, D_2)$; and the greatest common divisor of the two polynomials can be found by standard computer algebra techniques (see [13], section 4.1.2).

Open Question 2. Is there a completely effective method for proving all symmetric homogeneous inequalities?

7. TRIANGLE INEQUALITIES

The following result is well known ([27], [6], [28], [23]).

Theorem.
(The Fundamental Correspondence between Positive Inequalities and Triangle Inequalities)

Let a, b, c be the sides of a triangle. Then the inequality $f(a,b,c) \geq 0$ is equivalent to the inequality $f(x,y,z) \geq 0$ for all $x,y,z > 0$ where

$$x = b + c - a$$
$$y = c + a - b\,.$$
$$z = a + b - c$$

Variables a, b, and c can be expressed in terms of x, y, and z via the equations:

$$a = \frac{y+z}{2}$$
$$b = \frac{z+x}{2}\,.$$
$$c = \frac{x+y}{2}$$

Algorithm T (for Symmetric Homogeneous Triangle Inequalities). Given a proposed triangle inequality consisting of rational functions of a, b, and c, apply the substitution $a = (y+z)/2$, $b = (z+x)/2$, and $c = (x+y)/2$. Then apply algorithm R. The inequality may also involve the semiperimeter, s, in which case the substitution $s = (a+b+c)/2$ is made first.

Example 1.
Prove: $8(s-a)(s-b)(s-c) \leq abc$.
Solution: Letting $s = (a+b+c)/2$ and changing a, b, c to x, y, z gives

$$xyz \leq \frac{1}{8}\prod(x+y).$$

Expanding out into symmetric sums yields

$$6xyz \leq x^2y + xy^2 + y^2z + yz^2 + z^2x + x^2z$$

or

$$[1,1,1] \leq [2,1,0]$$

which is true by Muirhead's Theorem because $[1,1,1] \prec [2,1,0]$.

Example 2.
Prove: $64s^3\prod(s-a) \leq 27\prod a^2$.
Solution: Letting $s = (a+b+c)/2$ and changing a, b, c to x, y, z gives

$$27\sum x^4y^2 + 54\sum x^3y^3 \geq 10\sum x^4yz + 30\sum x^3y^2z + 114\sum x^2y^2z^2$$

which is equivalent to

$$27\sum! \, x^4y^2 + 27\sum! \, x^3y^3 \geq 5\sum! \, x^4yz + 30\sum! \, x^3y^2z + 19\sum! \, x^2y^2z^2$$

or

$$27[4,2,0] + 27[3,3,0] \geq 5[4,1,1] + 30[3,2,1] + 19[2,2,2]$$

which is true because

$$5[4,2,0] \succ 5[4,1,1]$$
$$22[4,2,0] \succ 22[3,2,1]$$
$$8[3,3,0] \succ 8[3,2,1]$$
$$19[3,3,0] \succ 19[2,2,2]$$

each of which follows from Muirhead's Theorem.

I tested Algorithm T on 23 examples from chapter 1 of Bottema [5]. Algorithm T successfully proved all 23 of the inequalities.

Theorem T1.

$$3\sum ab \leq \left(\sum a\right)^2 \quad (1)$$

$$\left(\sum a\right)^2 < 4\sum ab \quad (2)$$

$$\sum a^2 \geq \frac{36}{35}\left(s^2 + \frac{abc}{s}\right) \quad (3)$$

$$8\prod(s-a) \leq abc \quad (4)$$

$$8abc \leq \prod(a+b) \quad (5)$$

$$3\prod(a+b) \leq 8\sum a^3 \quad (6)$$

$$2\left(\sum a\right)\left(\sum a^2\right) \geq 3\left(\sum a^3 + 3abc\right) \quad (7)$$

$$abc < \sum a^2(s-a) \quad (8)$$

$$\sum a^2(s-a) \leq \frac{3}{2}abc \quad (9)$$

$$\sum ab(a+b) \geq 48\prod(s-a) \quad (10)$$

$$\sum a^3(s-a) \leq abcs \quad (11)$$

$$\sum a^5(s-a) \leq \frac{1}{2}abc\sum a^3 \quad (12)$$

$$64s^3\prod(s-a) \leq 27a^2b^2c^2 \quad (13)$$

$$\frac{2s}{abc} \leq \sum\frac{1}{a^2} \quad (14)$$

$$\sum\frac{1}{s-a} \geq \frac{9}{s} \quad (15)$$

$$\frac{3}{2} \leq \sum\frac{a}{b+c} \quad (16)$$

$$\sum\frac{a}{b+c} < 2 \quad (17)$$

$$\frac{15}{4} \leq \sum\frac{s+a}{b+c} \quad (18)$$

$$\sum\frac{s+a}{b+c} < \frac{9}{2} \quad (19)$$

$$\frac{1}{3} \leq \frac{\sum a^2}{(\sum a)^2} \qquad (20)$$

$$\frac{\sum a^2}{(\sum a)^2} < \frac{1}{2} \qquad (21)$$

$$\sum a^2 \sum b^3c^3 < 2\sum a^5 \sum a^3 \qquad (22)$$

$$\left(\sum a\right)^3 \leq 5\sum ab(a+b) - 3abc \qquad (23)$$

Computer Proof.
(1) is equivalent to $[2,0,0] \geq [1,1,0]$ which follows from Muirhead's Theorem.
(2) is equivalent to $2xy + 2yz + 2zx > 0$ which is true because a sum of positive terms is positive.
(3) is equivalent to $17[3,0,0] \geq 4[2,1,0] + 13[1,1,1]$ which follows from the following majorizations:

$$4[3,0,0] \succ 4[2,1,0]$$

$$13[3,0,0] \succ 13[1,1,1].$$

(4) is equivalent to $[2,1,0] \geq [1,1,1]$ which follows from Muirhead's Theorem.
(5) is equivalent to $[3,0,0] \geq [2,1,0]$ which follows from Muirhead's Theorem.
(6) is equivalent to $5[3,0,0]+3[2,1,0] \geq 8[1,1,1]$ which follows from the following majorizations:

$$3[2,1,0] \succ 3[1,1,1]$$

$$5[3,0,0] \succ 5[1,1,1].$$

(7) is equivalent to $[3,0,0] + [1,1,1] \geq 2[2,1,0]$ which follows from Schur's Inequality.
(8) is equivalent to $8xyz > 0$ which follows since a sum of positive terms is positive.
(9) is equivalent to $[2,1,0] \geq [1,1,1]$ which follows from Muirhead's Theorem.
(10) is equivalent to $[3,0,0]+5[2,1,0] \geq 6[1,1,1]$ which follows from the following majorizations:

$$[3,0,0] \succ [1,1,1]$$

$$5[2,1,0] \succ 5[1,1,1].$$

(11) is equivalent to $[2,2,0] \geq [2,1,1]$ which follows from Muirhead's Theorem.
(12) is equivalent to $5[4,2,0] + 3[3,3,0] + 3[2,2,2] \geq 5[4,1,1]+6[3,2,1]$ which follows from the following majorizations:

$$3[3,3,0] + 3[2,2,2] \geq 6[3,2,1]$$

$$5[4,2,0] \succ 5[4,1,1].$$

(13) is equivalent to $27[4,2,0] + 27[3,3,0] \geq 5[4,1,1] + 30[3,2,1] + 19[2,2,2]$ which follows from the following majorizations:

$$19[3,3,0] \succ 19[2,2,2]$$

$$8[3,3,0] \succ 8[3,2,1]$$

$$22[4,2,0] \succ 22[3,2,1]$$

$$5[4,2,0] \succ 5[4,1,1].$$

(14) is equivalent to $[4,0,0] \geq [2,2,0]$ which follows from Muirhead's Theorem.
(15) is equivalent to $[2,1,0] \geq [1,1,1]$ which follows from Muirhead's Theorem.
(16) is equivalent to $[3,0,0] + [2,1,0] \geq 2[1,1,1]$ which follows from the following majorizations:

$$[3,0,0] \succ [1,1,1]$$

$$[2,1,0] \succ [1,1,1].$$

(17) is equivalent to $3\sum! x^2y + 14xyz > 0$ which follows since a sum of positive terms is positive.
(18) is equivalent to $[3,0,0] + [2,1,0] \geq 2[1,1,1]$ which follows from the following majorizations:

$$[3,0,0] \succ [1,1,1]$$

$$[2,1,0] \succ [1,1,1].$$

(19) is equivalent to $9\sum! x^2y+42xyz > 0$ which follows since a sum of positive terms is positive.
(20) is equivalent to $[2,0,0] \geq [1,1,0]$ which follows from Muirhead's Theorem.
(21) is equivalent to $2xy + 2yz + 2zx > 0$ which follows since a sum of positive terms is positive.
(22) is equivalent to $6\sum x^8 + 24\sum! x^7y + 68\sum! x^6y^2 + 28\sum x^6yz + 120\sum! x^5y^3 + 36\sum! x^5y^2z + 142\sum x^4y^4 + 20\sum! x^4y^3z + 36\sum x^4y^2z^2 + 20\sum x^3y^3z^2 \geq 0$ which follows since a sum of positive terms is positive.
(23) is equivalent to $[3,0,0] + [1,1,1] \geq 2[2,1,0]$ which follows from Schur's Inequality.

8. OTHER PARTS OF A TRIANGLE

We can handle inequalities that reference other parts of a triangle (such as the area or the lengths of the medians) by expressing these parts in terms of a, b, and c.

For this purpose, the following formulae ([32]) can be used:

$$s = \frac{a+b+c}{2}$$

$$K = \sqrt{s(s-a)(s-b)(s-c)}$$

$$r = \frac{K}{s}$$

$$R = \frac{abc}{4K}$$

$$h_a = \frac{2K}{a}$$

$$m_a = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$$

$$t_a = \frac{2}{b+c}\sqrt{bcs(s-a)}$$

$$r_a = \frac{K}{s-a}$$

where s is the semiperimeter of the triangle, K is the area, r is the inradius, R is the circumradius, h_a is the length of the altitude to side a, m_a is the length of the median to side a, t_a is the length of the bisector of angle A, and r_a is the radius of the excircle relative to side a.

One problem with these substitutions is that they introduce square roots into the inequality. Algorithm T requires that both sides of the inequality be rational functions of a, b, and c. To get around this problem, we can try to express everything in terms of a, b, c, and K, where K is the area of the triangle. Then we isolate K to one side of the inequality and square both sides. This gets rid of all the radicals.

Algorithm K (for Symmetric Homogeneous Triangle Inequalities involving K).

Step 1. Express all of the elements of the triangle in terms of a, b, c, and K. If the result is not a rational function of a, b, c, and K, then this algorithm can not handle the inequality.

Step 2. Get rid of all denominators as in algorithm R. Each side of the inequality is now a polynomial in a, b, c, and K.

Step 3. Remove all even powers of K by applying the substitution $K^2 = s(s-a)(s-b)(s-c)$.

Step 4. Bring all terms involving K to one side of the inequality and bring all other terms to the other side. The inequality is now of the form $pK \geq q$ where p and q are symmetric homogeneous polynomials in a, b, and c.

Step 5. Apply Algorithm T to determine if p and q are positive. If $p \geq 0$ and $q \geq 0$ are both true, then go to step 6. If $p \leq 0$ and $q \leq 0$ are both true, then rewrite the inequality as $(-p)K \leq (-q)$ and go to step 6. If neither of those two cases hold, then this algorithm fails. [Note that if $p \geq 0$ is not a true inequality, it is not necessarily true that $p \leq 0$ is valid. It could happen that p is incomparable to 0. See Open Question 3 below for more details.]

Step 6. Square both sides of the inequality to get an equivalent inequality.

Step 7. Replace K^2 by $s(s-a)(s-b)(s-c)$.

Step 8. Apply algorithm T to determine the validity of this inequality.

I tested algorithm K using the following 17 examples from chapter 4 of Bottema [5]. The algorithm correctly proved 13 out of these 17 inequalities automatically.

Theorem T4.

$$s^2 \geq 3K\sqrt{3} \tag{1}$$

$$s^2 \geq 3K\sqrt{3} + \frac{1}{2}\sum(a-b)^2 \tag{2}$$

$$\sum a^2 \geq 4K\sqrt{3} \tag{3}$$

$$\sum ab \geq 4K\sqrt{3} \tag{4}$$

$$\sum ab \geq 4K\sqrt{3} + \frac{1}{2}\sum(a-b)^2 \tag{5}$$

$$4K\sqrt{3} + \sum(a-b)^2 \leq \sum a^2 \tag{6}$$

$$\sum a^2 \leq 4K\sqrt{3} + 3\sum(a-b)^2 \tag{7}$$

$$12K\sqrt{3} + 2\sum(a-b)^2 \leq \left(\sum a\right)^2 \tag{8}$$

$$\left(\sum a\right)^2 \leq 12K\sqrt{3} + 8\sum(a-b)^2 \tag{9}$$

$$\sum a^4 \geq 16K^2 \tag{10}$$

$$\sum a^4 \geq 16K^2 + 4K\sqrt{3}\sum(a-b)^2 + \frac{1}{2}\left(\sum(a-b)^2\right)^2 \tag{11}$$

$$\sum a^2b^2 \geq 16K^2 \tag{12}$$

$$4K\sqrt{3} \leq \frac{9abc}{\sum a} \tag{13}$$

$$(abc)^2 \geq (\frac{4K}{\sqrt{3}})^3 \tag{14}$$

$$\frac{1}{12}(\sum ab - \frac{1}{2}\sum a^2)(3\sum ab - \frac{5}{2}\sum a^2) \leq K^2 \tag{15}$$

$$K^2 \leq \frac{1}{12}(\sum ab - \frac{1}{2}\sum a^2)^2 \tag{16}$$

$$27\prod(b^2+c^2-a^2)^2 \leq (4K)^6 \tag{17}$$

Details of the computer proof of these theorems can be obtained from the author. Algorithm K failed to prove inequalities (7), (9), (11), and (17) automatically.

Other elements of a triangle, such as r and R, can be expressed in terms of K. I tested Algorithm K using the following 36 examples from chapter 5 of Bottema [5]. The algorithm correctly proved 32 out of these 36 inequalities automatically. The details are omitted. [Inequalities (3) and (25) were not proven due to a bug in the program and inequality (4) was keyed in wrong. Algorithm K could not prove inequality (8).]

Theorem T5.

$$2r \leq R \tag{1}$$

$$\sum a \leq 3R\sqrt{3} \tag{2}$$

$$s \leq 2R + (3\sqrt{3}-4)r \tag{3}$$

$$9r(4R+r) \leq 3s^2 \tag{4}$$

$$3s^2 \leq (4R+r)^2 \tag{5}$$

$$6r(4R+r) \leq 2s^2 \tag{6}$$

$$2s^2 \leq 2(2R+r)^2 + R^2 \tag{7}$$

$$2s^2(2R-r) \leq R(4R+r)^2 \tag{8}$$

$$r(16R-5r) \leq s^2 \tag{9}$$

$$s^2 \leq 4R^2 + 4Rr + 3r^2 \tag{10}$$

$$s^2 \geq 27r^2 \tag{11}$$

$$2s^2 \geq 27Rr \quad (12)$$

$$36r^2 \leq \sum a^2 \quad (13)$$

$$\sum a^2 \leq 9R^2 \quad (14)$$

$$24Rr - 12r^2 \leq \sum a^2 \quad (15)$$

$$\sum a^2 \leq 8R^2 + 4r^2 \quad (16)$$

$$36r^2 \leq \sum ab \quad (17)$$

$$\sum ab \leq 9R^2 \quad (18)$$

$$4r(5R - r) \leq \sum ab \quad (19)$$

$$\sum ab \leq 4(R + r)^2 \quad (20)$$

$$36r^2 \leq 4r(5R - r) \quad (21)$$

$$4r(5R - r) \leq \sum ab \quad (22)$$

$$4(R + r)^2 \leq 9R^2 \quad (23)$$

$$\sum a(s - a) \leq 9Rr \quad (24)$$

$$abc \leq 8R^2r + (12\sqrt{3} - 16)Rr^2 \quad (25)$$

$$\frac{\sqrt{3}}{R} \leq \sum \frac{1}{a} \quad (26)$$

$$\sum \frac{1}{a} \leq \frac{\sqrt{3}}{2r} \quad (27)$$

$$\frac{3\sqrt{3}}{2(R + r)} \leq \sum \frac{1}{a} \quad (28)$$

$$\frac{1}{R^2} \leq \sum \frac{1}{ab} \quad (29)$$

$$\sum \frac{1}{ab} \leq \frac{1}{4r^2} \quad (30)$$

$$8r(R - 2r) \leq \sum (a - b)^2 \quad (31)$$

$$\sum (a - b)^2 \leq 8R(R - 2r) \quad (32)$$

$$4r^2 \leq \frac{abc}{\sum a} \quad (33)$$

$$abc \leq (R\sqrt{3})^3 \quad (34)$$

$$5R - r \geq s\sqrt{3} \quad (35)$$

$$54Rr \leq 3\sum ab \quad (36)$$

The altitudes of a triangle can easily be expressed in terms of K. I tested Algorithm K using the following 14 examples from chapter 6 of Bottema [5]. The algorithm correctly proved 11 out of these 14 inequalities automatically.

Theorem T6.

$$2\sum h_a \leq \sqrt{3}\sum a \quad (1)$$

$$3\sum ab \geq \sum h_a h_b \quad (2)$$

$$\sum a^3 > \frac{8}{7}\sum h_a^3 \quad (3)$$

$$\sum \frac{a^2}{h_b^2 + h_c^2} \geq 2 \quad (4)$$

$$\sum h_a \geq 9r \quad (5)$$

$$\sum h_a \leq 3(R + r) \quad (6)$$

$$\sum h_a \leq 2R + 5r \quad (7)$$

$$\frac{2r(5R - r)}{R} \leq \sum h_a \quad (8)$$

$$\sum h_a \leq \frac{2(R + r)^2}{R} \quad (9)$$

$$2\sum h_a h_b \leq 6K\sqrt{3} \quad (10)$$

$$6K\sqrt{3} \leq 27Rr \quad (11)$$

$$\prod h_a \geq 27r^3 \quad (12)$$

$$\sum \frac{1}{h_a - 2r} \geq \frac{3}{r} \quad (13)$$

$$\sum \frac{h_a + r}{h_a - r} \geq 6 \quad (14)$$

Details of the computer proof are omitted. Algorithm K failed to prove inequalities (5), (8), and (12) automatically. For example, inequality (5) is equivalent to $2[6,0,0] + [4,1,1] + 5[3,3,0] + [2,2,2] \geq 4[5,1,0] + 3[4,2,0] + 2[3,2,1]$ which could not be handled.

It should be noted that the following heuristic can speed up Algorithm K considerably:

The GCD Heuristic. Given a proposed inequality of the form $P_1 \geq P_2$, if $g = \gcd(P_1, P_2)$ and if the inequality $g \geq 0$ can be proven to be true by Algorithm K, then we can divide both sides of the proposed inequality by g to get an equivalent inequality. [If $g \leq 0$ is true, then we can divide both sides by $(-g)$.]

Note that after dividing both sides by g, negative terms may be introduced, so it is necessary to collect like terms again and move the terms around so that only positive terms occur on each side.

In my computer program, I apply the GCD Heuristic after steps 2, 3, and 7 of Algorithm K.

Example.
Prove: $R \geq 2r$.
Solution by Algorithm K: Letting $r = \frac{K}{s}$ and $R = \frac{abc}{4K}$ yields $abc/4K \geq 2K/s$. The least common multiple of $4K$ and s is $4Ks$ and a recursive application of Algorithm K shows that $4Ks \geq 0$. We can thus multiply both sides of the inequality by $4Ks$ to get the equivalent inequality $abcs \geq 8K^2$. We now replace K^2 by $s(s-a)(s-b)(s-c)$. After expanding out, multiplying both sides by 2 and collecting terms, we get $a^4+b^4+c^4+a^2bc+b^2ca+c^2ab \geq 2a^2b^2+2b^2c^2+2c^2a^2$. Making the substitution $a=(y+z)/2$, $b=(z+x)/2$, $c=(x+y)/2$, gives, after multiplying both sides by 8 and collecting terms: $x^3y+xy^3+y^3z+yz^3+z^3x+zx^3+2x^2y^2+2y^2z^2+2z^2x^2 \geq 4x^2yz+4y^2zx+4z^2xy$. At this point, we can apply the GCD Heuristic, noting that the gcd of the left and right-hand sides is $x+y+z$. A quick check with Algorithm R shows that $x+y+z \geq 0$, so we can divide both sides of the inequality by $x+y+z$ to get $x^2y+xy^2+y^2z+yz^2+z^2x+zx^2-2xyz \geq 4xyz$. The point we wanted to illustrate is that now we have introduced a negative term on the left-hand side, so it is necessary to once again collect like terms. We do this and get $x^2y+xy^2+y^2z+yz^2+z^2x+zx^2 \geq 6xyz$. Expressing this result in terms of symmetric terms gives $[2,1,0] \geq [1,1,1]$ which is immediately seen to be true by Muirhead's Theorem since $[2,1,0] \succ [1,1,1]$.

Open Question 3. Is there a way to proceed when step 5 of Algorithm K fails?

For example, consider the inequality

$$\sum a^2 \leq 4K\sqrt{3} + 3\sum(a-b)^2.$$

When we isolate K to one side, we find that it's coefficient is $4\sqrt{3}$ which is clearly positive. However, the other side of the equation is $6\sum ab - 5\sum a^2$ which is not comparable to 0. In other words, this expression is sometimes positive and sometimes negative. Thus we can not square both sides of

$$4K\sqrt{3} \geq 6\sum ab - 5\sum a^2$$

to necessarily get an equivalent inequality. Hence we cannot get rid of the square root that will be introduced when we eliminate K. How else should we proceed then?

Inequalities involving the medians (or the angle bisectors) typically introduce 3 square roots into the inequality. If we could express everything in terms of one square root, then we could isolate this square root on one side of the inequality and then square both sides like we did for K. Unfortunately, this can not be done.

Theorem. There is no rational function, M, of a, b, and c such that each of m_a, m_b, and m_c can be expressed as rational functions of a, b, c, and $\sqrt{M}$.

Proof. Consider a 3-4-5 triangle. We have $m_a = \frac{1}{2}\sqrt{73}$ and $m_b = \sqrt{13}$. But $\sqrt{73}$ is not a member of the field $Q(\sqrt{13})$. Therefore no such M exists. □

9. TRIGONOMETRIC INEQUALITIES

If the inequality contains trigonometric functions of linear combinations of the angles of the triangle, such as $\sin(3A+B)$, then we can proceed as follows.

Step 1: Replace all occurrences of the functions cot, sec, and csc by the appropriate functions of tan, cos, and sin using the formulae:

$$\sec\theta = \frac{1}{\cos\theta}$$

$$\csc\theta = \frac{1}{\sin\theta}$$

$$\cot\theta = \frac{1}{\tan\theta}.$$

Step 2. Use the standard formulae for the sine, cosine, and tangent of a sum of angles to reduce the trigonometric expressions to functions of just A, B, and C (or $A/2$, $B/2$, $C/2$ if half-angles occur). If submultiples of the angles (other than 1/2) occur, then this algorithm will not handle the inequality.

Step 3. Replace the trigonometric expressions by their corresponding expressions involving a, b, c, s, and K so that algorithm K can be applied.

For this purpose, the following formulae ([32]) can be used:

$$\sin A = \frac{2K}{bc}$$

$$\cos A = \frac{b^2+c^2-a^2}{2bc}$$

$$\tan A = \frac{4K}{b^2+c^2-a^2}$$

$$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$$

$$\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$$

$$\tan\frac{A}{2} = \frac{K}{s(s-a)}$$

For a comprehensive treatment of the relationship between symmetric trigonometric expressions of the angles of a triangle, see Bager [2].

I tested algorithm K using the following 17 trigonometric examples from chapter 2 of Bottema [5]. The algorithm correctly proved all 17 of these inequalities automatically.

Theorem T2.

$$\sum \sin A \leq \frac{3}{2}\sqrt{3} \qquad (1)$$

$$\sum \sin^2 A \leq \frac{9}{4} \qquad (2)$$

$$\sum \sin A \geq \sum \sin 2A \qquad (3)$$

$$\prod \sin A \le \frac{3}{8}\sqrt{3} \tag{4}$$

$$1 < \sum \cos A \tag{5}$$

$$\sum \cos A \le \frac{3}{2} \tag{6}$$

$$\frac{3}{4} \le \sum \cos^2 A \tag{7}$$

$$\sum \cos^2 A < 3 \tag{8}$$

$$\sum \cos A \cos B \le \frac{3}{4} \tag{9}$$

$$\prod \cos A \le \frac{1}{8} \tag{10}$$

$$\prod \cos A \le \frac{1}{24} \sum \cos^2(A-B) \tag{11}$$

$$\sum \cot A \ge \sqrt{3} \tag{12}$$

$$\sum \cot^2 A \ge 1 \tag{13}$$

$$\sum \csc A \ge 2\sqrt{3} \tag{14}$$

$$\sum \csc^2 A \ge 4 \tag{15}$$

$$\frac{1 + \prod \cos A}{\prod \sin A} \ge \sqrt{3} \tag{16}$$

$$2\sum \cot A \ge \sum \csc A \tag{17}$$

Details of the computer proof are omitted.

Another problem we have encountered when trying to prove triangle inequalities, is that not all expressions can be written as rational functions of a, b, c, and K.

Theorem. The expressions $\sum \sin \frac{A}{2}$ and $\sum \cos \frac{A}{2}$ can not be expressed as rational functions of a, b, c, and K.

Proof. Consider a 3-4-5 triangle. We find that $\sum \sin \frac{A}{2}$ and $\sum \cos \frac{A}{2}$ involve radicals. But a, b, c, and K are rational; so these expressions cannot be rational functions of a, b, c, and K. □

10. A HEURISTIC FOR HANDLING MULTIPLE SQUARE ROOTS

Cauchy's Inequality.

$$\left(\sum_{i=1}^{n} x_i y_i\right)^2 \le \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right).$$

For a proof see [15], page 16.

If we let $y_i = 1$ in Cauchy's inequality, we get:

$$\left(\sum_{i=1}^{n} x_i\right)^2 \le n\left(\sum_{i=1}^{n} x_i^2\right).$$

We may use this inequality to get rid of square root signs in certain inequalities. Note that for triangle inequalities, $n = 3$.

Square Root Heuristic. If an inequality is of the form $\sum \sqrt{f(x)} \le g(x)$, then apply the inequality

$$\sum \sqrt{f(x)} \le \sqrt{3\sum f(x)}$$

to see if $\sqrt{3\sum f(x)} \le g(x)$ or equivalently $\sum f(x) \le \frac{1}{3}g(x)^2$.

Example 1.

Prove: $\sum \sqrt{\sin A} \le 3\sqrt[4]{3/4}$.

Solution: We note that by the square root heuristic

$$\sum \sqrt{\sin A} \le \sqrt{3\sum \sin A}.$$

But it is known that

$$\sum \sin A \le \frac{3}{2}\sqrt{3},$$

so therefore

$$\sum \sqrt{\sin A} \le \sqrt{3(\frac{3}{2}\sqrt{3})} = 3\sqrt[4]{\frac{3}{4}}.$$

Example 2.

Prove: $\sum \sqrt{h_a} \le \frac{3}{2}\sqrt{6R}$.

Solution: We note that by the square root heuristic

$$\sum \sqrt{h_a} \le \sqrt{3\sum h_a}.$$

But our algorithm can prove that

$$\sum h_a \le \frac{9}{2}R,$$

so therefore

$$\sum \sqrt{h_a} \le \sqrt{3\sum h_a} \le \sqrt{3\frac{9}{2}R} = \frac{3}{2}\sqrt{6R}.$$

11. SUMMARY

It should be noted that the way I coded Algorithm K, not only can it decide if an inequality is true, but if the inequality is true it will print out a proof of the inequality that a human can follow.

I have tested Algorithm K against many inequalities from Bottema [5]. The following table summarizes the current state of success.

Section number	Number of inequalities	Number proven	Success rate
1	23	23	100%
2	17	17	100%
3	6	2	33%
4	17	13	76%
5	36	32	89%
6	14	11	79%
7	15	11	73%
total	128	109	85%

Note that only symmetric homogeneous triangle inequalities were selected for testing, and no inequalities involving medians, angle bisectors, exradii, or half-angles were chosen. Computing times (using an Alliant FX/80 with one processor) varied from 15 seconds to 3 minutes, depending on the complexity of the inequality.

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