Some Metric Inequalities for Lattice Polygons

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ABSTRACT

Given a convex lattice polygon with g interior lattice points, we find upper and lower bounds for the perimeter, diameter, and width of the polygon. For small g, the extremal figures were found by computer.

A lattice point in the plane is a point with integer coordinates. The set of all lattice points in the plane is denoted by Z^2. A lattice polygon is a simple polygon whose vertices are all lattice points.

In this article, we will discuss inequalities for convex lattice polygons containing g interior lattice points. With g fixed, we will find upper and lower bounds for the perimeter P, diameter D, and width w, of the lattice polygon.

1. Inequalities for Perimeter

Let P denote the perimeter of the lattice polygon.

Proposition 1.1 (P unbounded). *For any positive integer, g, there are convex lattice polygons with g interior lattice points and arbitrarily large perimeter.*

Proof. Consider the lattice triangle OAB where $O = (0,0)$, $A = (2g+2, 0)$, and $B = (0,2)$. Applying a shear of magnitude k, leaving the x-axis fixed, namely,

$$x' = x + ky$$
$$y' = y$$

where $k \in Z$, we find that O and A remain fixed, and B moves to $(2k, 2)$. This triangle has g interior lattice points and has a perimeter larger than $OB = \sqrt{4k^2+4}$. But we can make $\sqrt{4k^2+4}$ arbitrarily large by making k arbitrarily large. Thus, P is unbounded. $\square$

Two polygons are said to be *lattice congruent* if they are congruent and each is a lattice polygon.

The author investigated by computer the relationship between P and g for all convex lattice polygons with $D \leq 10$ and $g \leq 10$. In each case, the minimum value of P was found, which was always less than 14. For details of the computer search, see [4].

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Proposition 1.2 (effectiveness of search for minimum P). *No lattice polygons with $P \le 14$ were missed by the computer search.*

Proof. Consider any polygon with $g \le 10$ and $P \le 14$. Then using the fact that in any convex body in the plane, $D \le P/2$, we would have $D \le 7$. Thus the polygon would have been found in our search since we searched all polygons with $D \le 10$. $\square$

The following results were found by computer:

Proposition 1.3. *If $g = 0$ then $P \ge 2 + \sqrt{2} \approx 3.414$. Equality occurs when and only when K is lattice congruent to the isosceles right triangle with vertices $(0,0)$, $(0,1)$, $(1,0)$. See figure 1-1.*

```
o .
o o
```

Figure 1-1
Unique polygon with g=0
and smallest perimeter

Proposition 1.4. *If $g = 1$ then $P \ge 4\sqrt{2} \approx 5.657$. Equality occurs when and only when K is lattice congruent to the diamond with vertices $(0,1)$, $(1,0)$, $(1,2)$, $(2,1)$. See figure 1-2.*

```
. o .
o . o
. o .
```

Figure 1-2
Unique polygon with g=1
and smallest perimeter

Proposition 1.5. *If $g = 2$ then $P \ge 2\sqrt{2} + 2\sqrt{5} \approx 7.301$. Equality occurs when and only when K is lattice congruent to the kite with vertices $(0,1)$, $(1,0)$, $(1,2)$, $(3,1)$ or to the parallelogram with vertices $(0,1)$, $(1,0)$, $(2,2)$, $(3,1)$. See figure 1-3.*

```
. o . .        . . o .
o . . o        o . . o
. o . .        . o . .
```

Figure 1-3
Only polygons with g=2
and smallest perimeter

Proposition 1.6. *If $g = 3$ then $P \ge 3\sqrt{2} + 2\sqrt{5} \approx 8.715$. Equality occurs when and only when K is lattice congruent to the quadrilateral with vertices $(0,1)$, $(1,3)$, $(2,0)$, $(3,1)$ or to the quadrilateral with vertices $(0,1)$, $(1,3)$, $(1,0)$, $(3,1)$. See figure 1-4.*

```
. o . .        . o . .
. . . .        . . . .
o . . o        o . . o
. . o .        . o . .
```

Figure 1-4
Only polygons with g=3
and smallest perimeter

Proposition 1.7. *If $g = 4$ then $P \geq 4\sqrt{5} \approx 8.944$. Equality occurs when and only when K is lattice congruent to the square with vertices $(0,1)$, $(2,0)$, $(1,3)$, $(3,2)$. See figure 1-5.*

```
. o . .
. . . o
o . . .
. . o .
```

Figure 1-5
Unique polygon with g=4
and smallest perimeter

Proposition 1.8. *If $g = 5$ then $P \geq 3\sqrt{5}+\sqrt{13} \approx 10.314$. Equality occurs when and only when K is lattice congruent to the quadrilateral with vertices $(0,2)$, $(2,3)$, $(3,0)$, $(4,2)$. See figure 1-6.*

```
. . o . .
o . . . o
. . . . .
. . . o .
```

Figure 1-6
Unique polygon with g=5
and smallest perimeter

Proposition 1.9. *If $g = 6$ then $P \geq 2\sqrt{5} + 2\sqrt{10} \approx 10.797$. Equality occurs when and only when K is lattice congruent to the parallelogram with vertices $(0,1)$, $(1,3)$, $(3,0)$, $(4,2)$. See figure 1-7.*

```
. o . . .
. . . . o
o . . . .
. . . o .
```

Figure 1-7
Unique polygon with g=6
and smallest perimeter

Proposition 1.10. *If $g = 7$ then $P \geq 2\sqrt{5} + 2\sqrt{13} \approx 11.683$. Equality occurs when and only when K is lattice congruent to the kite with vertices $(0,1)$, $(2,0)$, $(2,4)$, $(4,1)$ or to* *the parallelogram with vertices* $(0,2)$, $(1,4)$, $(3,0)$, $(4,2)$. *See figure 1-8.*

```
. . o . .
. . . . .       . o . . .
. . . . .       . . . . .
o . . . o       o . . . o
. . o . .       . . . . .
                . . . o .
```

Figure 1-8
Only polygons with g=7
and smallest perimeter

Proposition 1.11. *If* $g = 8$ *then* $P \geq 4\sqrt{5} + \sqrt{10} \approx 12.107$. *Equality occurs when and only when* K *is lattice congruent to the pentagon with vertices* $(0,1)$, $(1,3)$, $(3,4)$, $(4,1)$, $(2,0)$. *See figure 1-9.*

```
. . . o .
. o . . .
. . . . .
o . . . o
. . o . .
```

Figure 1-9
Unique polygon with g=8
and smallest perimeter

Proposition 1.12. *If* $g = 9$ *then* $P \geq 4\sqrt{10} \approx 12.649$. *Equality occurs when and only when* K *is lattice congruent to the square with coordinates* $(0,1)$, $(1,4)$, $(4,3)$, $(3,0)$. *See figure 1-10.*

```
. o . . .
. . . . o
. . . . .
o . . . .
. . . o .
```

Figure 1-10
Unique polygon with g=9
and smallest perimeter

Proposition 1.13. *If* $g = 10$ *then* $P \geq 6\sqrt{5} \approx 13.416$. *Equality occurs when and only when* K *is lattice congruent to the equilateral hexagon with vertices* $(0,1)$, $(1,3)$, $(3,4)$, $(5,3)$, $(4,1)$, $(2,0)$. *See figure 1-11.*

```
. . . o . .
. o . . . o
. . . . . .
o . . . o .
. . o . . .
```

Figure 1-11
Unique polygon with g=10
and smallest perimeter

A general result can also be obtained, although this result is not best possible.

Theorem 1.14. $P > \sqrt{(4g+2)\pi}$.

Proof. Pick's Formula (see [2]) says that for a lattice polygon with area A, g interior lattice points, and b boundary lattice points, $A = b/2 + g - 1$. Since $b \geq 3$, we see that $A \geq g + 1/2$. Combining this result with the isoperimetric inequality, $P^2 \geq 4\pi A$, (see [1], p. 117) gives

$$P^2 \geq 4\pi A \geq 4\pi(g + \frac{1}{2})$$

or taking square roots,

$$P \geq \sqrt{(4g+2)\pi}.$$

$\square$

Proposition 1.15. $P \geq 2\lceil\sqrt{g}\,\rceil + 2$.

This will be proven in the next section (corollary 2.15).

The Enclosed Square Lemma. *Let g be a positive integer. Then there is a convex lattice polygon containing exactly g interior lattice points and bounded by a square of side $\lceil\sqrt{g}\,\rceil + 1$.*

Proof. Suppose that g is between $n^2 + 1$ and $(n+1)^2$ inclusive, so that $\lceil\sqrt{g}\,\rceil = n + 1$.

If $g = (n+1)^2$, then we may take the square with side length $n + 2$ whose sides are parallel to the axes. This contains exactly g interior lattice points.

Keeping the outer square $ABCD$ fixed, we wish to show that we can remove interior lattice points by making the enclosing convex lattice polygon smaller. The number of lattice points we need remove varies from 0 to $2n$ since $(n+1)^2 - 2n = n^2 + 1$. It suffices to show that we can remove anywhere from 0 to n lattice points near edges AB and BC or near vertex B, not including the interior lattice points closest to vertices A and C. For then we can apply the same process around vertex D, along edges DA and DC.

Consider point B to be the origin. To remove one lattice point, move the vertex at $(0,0)$ to $(1,1)$. To remove two lattice points, choose vertices at $(2,0)$ and $(0,4)$ instead. To remove three lattice points, choose as vertices $(3,0)$ and $(0,3)$. To remove four lattice points, choose $(2,1)$ and $(0,5)$ as vertices. To remove five lattice points, choose $(2,1)$ and $(0,7)$ as vertices. To remove six lattice points, choose $(4,0)$ and $(0,4)$ as lattice points. (Note: this assumes $n > 6$. Smaller values of n can easily be handled as special cases.)

Finally, to remove from 7 to n lattice points, choose as vertices $(2,2)$, $(2k+2,0)$, and $(0,2j+2)$ where

$$1 \leq j \leq \lfloor \frac{n}{2} \rfloor$$
$$1 \leq k \leq \lfloor \frac{n}{2} \rfloor$$

which removes 4+j+k lattice points. As j and k are varied in their allowable ranges, from 7 to n lattice points can be removed. $\square$

Proposition 1.16. *Let $P(K)$ denote the perimeter of lattice polygon K. Let $P(g) = \min\{P(K) | g(K) = g\}$. Then $P(g) \leq 4(\lceil \sqrt{g} \rceil + 1)$.*

Proof. This follows from the Enclosed Square Lemma which guarantees a polygon containing g lattice points and enclosed inside a square of side $\lceil \sqrt{g} \rceil + 1$. The polygon must have perimeter less than the perimeter of this square (since both are convex). $\square$

We may summarize this data as follows:

Theorem 1.17. *Let $P(K)$ denote the perimeter of the convex lattice polygon K. Let $P(g) = \min\{P(K) | g(K) = g\}$ where $g(K)$ denotes the number of lattice points in the interior of K. Then*

a. $P(0) = 2 + \sqrt{2}$.
b. $P(1) = 4\sqrt{2}$.
c. $P(2) = 2\sqrt{2} + 2\sqrt{5}$.
d. $P(3) = 3\sqrt{2} + 2\sqrt{5}$.
e. $P(4) = 4\sqrt{5}$.
f. $P(5) = 3\sqrt{5} + \sqrt{13}$.
g. $P(6) = 2\sqrt{5} + 2\sqrt{10}$.
h. $P(7) = 2\sqrt{5} + 2\sqrt{13}$.
i. $P(8) = 4\sqrt{5} + \sqrt{10}$.
j. $P(9) = 4\sqrt{10}$.
k. $P(10) = 6\sqrt{5}$.
l. $P(g) > \sqrt{(4g+2)\pi}$.
m. $2(\lceil \sqrt{g} \rceil + 1) \leq P(g) \leq 4(\lceil \sqrt{g} \rceil + 1)$.

2. Inequalities for Diameter

Let D denote the diameter of a convex lattice polygon, K.

Proposition 2.1 (D unbounded). *For any positive integer, g, there are convex lattice polygons with g interior lattice points and arbitrarily large diameter.*

Proof. Consider the lattice triangle OAB where $O = (0,0)$, $A = (2g+2, 0)$, and $B = (0,2)$. Applying a shear of magnitude k, leaving the x-axis fixed, we find that O and A remain fixed, and B moves to $(2k, 2)$. This triangle has g interior lattice points and has a diameter at least as large as $OB = \sqrt{4k^2+4}$. But we can make $\sqrt{4k^2+4}$ arbitrarily large by making k arbitrarily large. Thus, D is unbounded. $\square$

The author investigated by computer the relationship between D and g for all convex lattice polygons with $D \le 10$ and $g \le 10$. In each case, the minimum value of D was found, which was always less than 5. No lattice polygons with $D \le 5$ were missed by the computer search because all lattice polygons with $D \le 10$ were examined.

The following results were found by computer:

Proposition 2.2. *If $g = 0$ then $D \ge \sqrt{2}$. Equality occurs when and only when K is lattice congruent to the square with vertices $(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$ or to the isosceles right triangle with vertices $(0,0)$, $(0,1)$, $(1,0)$. See figure 2-1.*

```
o o        o .
o o        o o
```

Figure 2-1
Only polygons with g=0
and smallest diameter

Proposition 2.3. *If $g = 1$ then $D \ge 2$. Equality occurs when and only when K is lattice congruent to the diamond with vertices $(0,1)$, $(1,0)$, $(2,1)$, $(1,2)$. See figure 2-2.*

```
. o .
o . o
. o .
```

Figure 2-2
Unique polygon with g=1
and smallest diameter

Proposition 2.4. *If $g = 2$ then $D \ge 3$. Equality occurs when and only when K is lattice congruent to one of the figures shown below.*

```
o . . o        . o-o .
. . . .        o . . o
. o o .        . o-o .
```

Figure 2-3
Only polygons with g=2
and smallest diameter

A pair of circles connected by a dash means that the polygon must contain one or both of these lattice points as vertices.

Proposition 2.5. *If $g = 3$ then $D \ge 3$. Equality occurs when and only when K is lattice congruent to the trapezoid with vertices $(0,1)$, $(1,0)$, $(1,3)$, $(3,1)$. See figure 2-4.*

```
. o . .
. . . .
o . . o
. o . .
```

Figure 2-4
Unique polygon with g=3
and smallest diameter

Proposition 2.6. *If $g = 4$ then $D \geq \sqrt{10}$. Equality occurs when and only when K is lattice congruent to a polygon whose vertices consist of a subset of the vertices of the octagon pictured below. Remove any 0, 1, 2, 3, or 4 of its vertices, but never remove two consecutive vertices.*

```
. o o .
o . . o
o . . o
. o o .
```

Figure 2-5
Polygon with g=4
and smallest diameter

Proposition 2.7. *If $g = 5$ then $D \geq 4$. Equality occurs when and only when K is lattice congruent to one of the polygons pictured below.*

```
. . o . .
. . . . .      . o x . .
o . . . o      . . . x .
. . . . .      o . . . o
. . o . .      . o-o-o .
```

Figure 2-6
Only polygons with g=5
and smallest diameter

A set of three circles connected by dashes means that some non-empty subset of these three vertices must be vertices of the polygon. A lattice point marked with an x represents an optional point; it may or may not belong to the polygon.

Proposition 2.8. *If $g = 6$ then $D \geq 4$. Equality occurs when and only when K is lattice congruent to one of the polygons pictured below.*

```
. o . o .
. . . . .
o . . . o
. o-o-o .
```

Figure 2-7
Only polygons with g=6
and smallest diameter

A set of three circles connected by dashes means that some non-empty subset of these three vertices must be vertices of the polygon.

Proposition 2.9. *If $g = 7$ then $D \geq 4$. Equality occurs when and only when K is lattice congruent to one of the polygons pictured below.*

```
. . o . .
. x . x .
. . . . .
o . . . o
. . o . .
```

Figure 2-8
Only polygons with g=7
and smallest diameter

A lattice point marked with an x represents an optional point; it may or may not belong to the polygon.

Proposition 2.10. *If $g = 8$ then $D \geq 3\sqrt{2}$. Equality can hold as can be seen by figure 2-9 in which $g = 8$ and $D = 3\sqrt{2}$.*

```
. o . . .
. . . o .
. . . . .
o . . . o
o o . . .
```

Figure 2-9
Polygon with g=8
and smallest diameter

There were too many figures in which equality held to warrant listing them all here.

Proposition 2.11. *If $g = 9$ then $D \geq 3\sqrt{2}$. Equality occurs when and only when K is lattice congruent to one of the polygons pictured below.*

```
. o . o .
. . . . .
x . . . x
o . . . o
. . o . .
```

Figure 2-10
Only polygons with g=9
and smallest diameter

A lattice point marked with an x represents an optional point; it may or may not belong to the polygon.

Proposition 2.12. *If $g = 10$ then $D \geq 5$. Equality can hold as can be seen by figure 2-11 in which $g = 10$ and $D = 5$.*

```
. . . o . .
. . . . . .
o . . . . o
. . . . . .
. . . . . .
. . . o . .
```

Figure 2-11
Polygon with g=10
and smallest diameter

There were too many figures in which equality held to warrant listing them all here.

A general result can also be obtained, although this result is not best possible.

Lemma 2.13. *Let K be a convex body and let $H = \mathrm{hull}(K^\circ \cap Z^2)$ where K° denotes the interior of K. Let K_x and H_x denote the horizontal width of K and H respectively. Then $K_x \geq H_x + 2$.*

This is reasonably obvious after projecting K down to the x-axis.

Theorem 2.14. $D \geq \lceil \sqrt{g}\, \rceil + 1$.

Proof. Let $s = \lceil \sqrt{g}\, \rceil - 1 = \lfloor \sqrt{g-1} \rfloor$, so that

$$g > s^2. \tag{1}$$

Let H be the convex hull of $K^\circ \cap Z^2$. Let H_x denote the horizontal width of H and let H_y denote the vertical width of H. From (1) we see that either H_x or H_y must be larger than s, for if both H_x and H_y were less than or equal to s, then H could be covered by the square of side $s-1$ (consisting of s^2 lattice points) and we would have $g \leq s^2$.

Thus we may assume without loss of generality that $H_x > s$. Then by lemma 2.13, we would have $K_x \geq H_x + 2$. Thus $K_x > s + 2$. But the diameter of a convex body must be at least as large as the horizontal width (after all, the diameter is the largest of all the directional widths), so we see that $D \geq s + 2$. □

Corollary 2.15. $P \geq 2\lceil \sqrt{g}\, \rceil + 2$.

This follows from the fact that P is always greater than $2D$.

Proposition 2.16. *There is a convex lattice polygon with diameter satisfying $D \leq (\lceil \sqrt{g}\, \rceil + 1)\sqrt{2}$.*

Proof. By the Enclosed Square Lemma, we can find a lattice polygon with g interior lattice points inside a square of side $t = \lceil \sqrt{g}\, \rceil + 1$. The diameter of this polygon must be smaller than the diameter of the enclosed square, which is $t\sqrt{2}$. □

Notation. Let $D(g) = \min\{D(K) | g(K) = g\}$ where $D(K)$ denotes the diameter of the convex polygon K.

Proposition 2.17. $D(10) = D(11) = D(12) = D(13) = 5$.

Proof. We have already seen that $D(g) \geq \lceil \sqrt{g}\, \rceil + 1$ so that $D(g) \geq 5$ if $g = 10, 11, 12$, or 13. To prove that $D(g) = 5$ in these cases, it is only necessary to exhibit a lattice polygon with diameter 5 for these cases. Figure 2-11 already established this for $g = 10$. We conclude the proof by exhibiting lattice polygons with diameter 5 in figure 2-12. $\square$

```
                                . . . . o .
. o . o . .     . o . . o .     . . . . . .
. . . . . .     . . . . . .     o . . . . o
o . . . . o     o . . . . o     . . . . . .
. . . . . .     . . . . . .     . o . . . .
. o . . o .     . o . . o .     . . . . o .
```

Figure 2-12
Polygons with g=11, 12, and 13
and smallest diameter

Proposition 2.18. $D(17) = D(18) = D(19) = D(20) = D(21) = 6$.

Proof. Again, we need only exhibit the appropriate lattice polygons with diameter 6. See figure 2-13. $\square$

```
. . o . . . .     . . o . . .      . . o . . . .
. . . . . . .     . . . . o .      . . . . . . .
. . . . . . .     o . . . . .      . . . . . . .
o . . . . . o     . . . . . .      o . . . . . o
. . . . . . .     . . . . . .      . . . . . . .
. . . . . . .     o . . . . o      . . . . . o .
. . o . . . .     . . o . . .      . . o . . . .


. . o . . . .     . . o . . . .
. o . . . o .     . . . . . o .
. . . . . . .     . . . . . . .
. . . . . . .     o . . . . . o
o . . . . . o     . . . . . . .
. . . . . . .     . . . . . o .
. . o . . . .     . . o . . . .
```

Figure 2-13
Polygons with g=17, 18, 19, 20, and 21
and smallest diameter

Proposition 2.19. $D(26) = D(27) = D(28) = 7$.

Proof. Figure 2-14 shows lattice polygons with appropriate g and $D = 7$. Since we have already shown $D(g) \geq 7$ for g in this range, this completes the proof. $\square$

```
. . o . . . .      . . . o . . . .      . . . o . . . .
. . . . . o .      . . . . . o . .      . . . . . o . .
o . . . . . .      . o . . . . . .      . . . . . . . .
. . . . . . .      . . . . . . . .      o . . . . . . o
. . . . . . o      . . . . . . . .      . . . . . . . .
o . . . . . .      o . . . . . . o      . . . . . . . .
. . . . . o .      . o . . . . o .      . o . . . . o .
. . o . . . .      . . . o . . . .      . . . o . . . .
```

Figure 2-14
Polygons with g=26, 27, and 28
and smallest diameter

Proposition 2.20. $D(37) = 8$.

Proof. Figure 2-15 shows a lattice polygon with $g = 37$ and $D = 8$. Since we have already shown $D(37) \geq 8$, this completes the proof. $\square$

```
. . o . . . . . .
. . . . . . . o .
. . . . . . . . .
. . . . . . . . .
o . . . . . . . o
. . . . . . . . .
. o . . . . . o .
. . o . . . . . .
```

Figure 2-15
Polygon with g=37
and smallest diameter

We may summarize this data as follows:

Theorem 2.21. *Let $D(K)$ denote the diameter of the convex lattice polygon K. Let $D(g) = \min\{D(K)|g(K) = g\}$ where $g(K)$ denotes the number of lattice points in the interior of K. Then*

a. $D(0) = \sqrt{2}$.
b. $D(1) = 2$.
c. $D(2) = 3$.
d. $D(3) = 3$.
e. $D(4) = \sqrt{10}$.
f. $D(5) = 4$.
g. $D(6) = 4$.
h. $D(7) = 4$.
i. $D(8) = 3\sqrt{2}$.
j. $D(9) = 3\sqrt{2}$.
k. $D(10) = 5$.
l. $D(11) = 5$.

m. $D(12) = 5$.
n. $D(13) = 5$.
o. $D(17) = 6$.
p. $D(18) = 6$.
q. $D(19) = 6$.
r. $D(20) = 6$.
s. $D(21) = 6$.
t. $D(26) = 7$.
u. $D(27) = 7$.
v. $D(28) = 7$.
w. $D(37) = 8$.
x. $\lceil\sqrt{g}\,\rceil + 1 \le D(g) \le (\lceil\sqrt{g}\,\rceil + 1)\sqrt{2}$.

3. Inequalities for minimal width

Let w denote the (minimal) width of a convex lattice polygon, K.

An *altitude* of a polygon is a line through a vertex and perpendicular to a side of the polygon not incident with that vertex. The length of the altitude is the distance from the vertex to the foot of the perpendicular.

Note that the foot of the perpendicular may lie outside the polygon, on the extension of the side to which the altitude is drawn.

Algorithm 3.1 (Computation of the width). *The width of a convex polygon can be computed by going to each vertex and finding the length of the largest altitude emenating from that vertex. The width of the polygon is then the smallest of these altitudes.*

The verification of this is straightforward. This provides an effective means for computing the width of a polygon.

Proposition 3.2 (Minimal w). *For any positive integer, g, there are convex lattice polygons with g interior lattice points and width arbitrarily small.*

This follows from the fact that D can be made arbitrarily large (while preserving the area) and the inequality $wD \le 2A$ which holds for all convex bodies in the plane (see [1], p. 87).

The author investigated by computer the relationship between w and g for all convex lattice polygons with $D \le 10$ and $g \le 10$. In each case, the maximum value of w was found, which was always less than 5.

Proposition 3.3 (effectiveness of search for maximum w). *No lattice polygons with $w \ge 5$ were missed by the computer search.*

Proof. Consider any polygon with $g \le 10$ and $w \ge 5$. We use the fact that $wD \le 2A$ for all convex bodies in the plane (see [1], p. 87), and that if a lattice polygon has at least one interior lattice point, then $b \le 2g + 7$ (see [5]). We also use Pick's Formula ([2]), $A = b/2 + g - 1$. Then, for $g > 0$,

$$D \le \frac{2A}{w} = \frac{b + 2g - 2}{w} \le \frac{4g + 5}{w} \le \frac{45}{5} \le 9.$$

Thus the polygon would have been found in our search since we searched all polygons with $D \leq 10$. (We handle $g = 0$ as a special case.) $\square$

The following results were found by computer:

Proposition 3.4. *If $g = 0$ then $w \leq \sqrt{2} \approx 1.414$. Equality occurs when and only when K is lattice congruent to the isosceles right triangle with vertices $(0,0)$, $(0,2)$, $(2,0)$. See figure 3-1.*

```
o . .
. . .
o . o
```

Figure 3-1
Unique polygon with g=0
and largest width

Proposition 3.5. *If $g = 1$ then $w \leq 3\sqrt{2}/2 \approx 2.121$. Equality occurs when and only when K is lattice congruent to the isosceles right triangle with vertices $(0,0)$, $(0,3)$, and $(3,0)$. See figure 3-2.*

```
o . . .
. . . .
. . . .
o . . o
```

Figure 3-2
Unique polygon with g=1
and largest width

Proposition 3.6. *If $g = 2$ then $w \leq 2$. Equality can hold as can be seen by the first polygon in figure 3-3 in which $g = 2$ and $w = 2$.*

```
                 o . . . .
                 . . . . .          o . . o
o . . o          . . . . .          . . . .
. . . .          . . . . .          . . . .
o . . o          o . . . o          o . . o
```

Figure 3-3
Polygons with g=2, g=3, g=4 and largest width

Proposition 3.7. *If $g = 3$ then $w \leq 2\sqrt{2} \approx 2.828$. Equality can hold as can be seen by the triangle in figure 3-3 in which $g = 3$ and $w = 2\sqrt{2}$.*

Proposition 3.8. *If $g = 4$ then $w \leq 3$. Equality can hold as can be seen by the last polygon in figure 3-3 in which $g = 4$ and $w = 3$.*

Proposition 3.9. *If $g = 5$ then $w \leq 8\sqrt{5}/5 \approx 3.578$. Equality occurs when and only when K is lattice congruent to the triangle with vertices $(0,0)$, $(4,0)$, $(2,4)$. See figure 3-4.*

```
. . o . .
. . . . .
. . . . .
. . . . .
o . . . o
```

Figure 3-4
Only polygon with g=5
and largest width

Proposition 3.10. *If $g = 6$ then $w \leq 8\sqrt{5}/5 \approx 3.578$. Equality can hold as can be seen by the figure below in which $g = 6$ and $w = 8\sqrt{5}/5$.*

```
. . o . .
. . . . .
. . . . .
o . . . .
o . . . o
```

Figure 3-5
Polygon with g=6
and largest width

Proposition 3.11. *If $g = 7$ then $w \leq 4$. Equality can hold as can be seen by the figure below in which $g = 7$ and $w = 4$.*

```
. . o . .
. . . . .
o . . . o
. . . . .
o . . . o
```

Figure 3-6
Polygon with g=7
and largest width

Proposition 3.12. *If $g = 8$ then $w \leq 4$. Equality can hold as can be seen by the figure below in which $g = 8$ and $w = 4$.*

```
o . o . .
. . . . .
. . . . o
. . . . .
o . . . o
```

Figure 3-7
Polygon with g=8
and largest width

Proposition 3.13. *If $g = 9$ then $w \leq 9\sqrt{5}/5 \approx 4.025$. Equality can hold as can be seen by the figure below in which $g = 9$ and $w = 9\sqrt{5}/5$.*

```
. . o . . .
. . . . . .
. . . . . .
. . . . . o
o . . . . .
o o . . . .
```

Figure 3-8
Polygon with g=9
and largest width

Proposition 3.14. *If $g = 10$ then $w \leq 2\sqrt{5} \approx 4.472$. Equality can hold as can be seen by the figure below in which $g = 10$ and $w = 2\sqrt{5}$.*

```
. . o . . .
. . . o . .
. . . . . .
. . . . . .
. . . . . .
o . . . . o
```

Figure 3-9
Polygon with g=10 and largest width

A general result can also be obtained, although this result is not best possible.

Theorem 3.15. $w \leq \sqrt{2(g+1)\sqrt{3}}$.

Proof. For $g = 0$ and $g = 1$, this result is true by Propositions 3.4 and 3.5. If $g > 1$, Scott [5] has shown that $b \leq 2g + 6$. Combining this with Pick's Formula, $A = b/2 + g - 1$, gives $A \leq 2g + 2$. We can combine this with the inequality $w^2 \leq A\sqrt{3}$ which is true for all convex bodies (see [1], p. 83), to get $w^2 \leq (2g+2)\sqrt{3}$. Taking square roots of both sides gives us the desired result. □

We may summarize this data as follows:

Theorem 3.16. *Let $w(K)$ denote the width of the convex lattice polygon K. Let $w(g) = \max\{w(K) | g(K) = g\}$ where $g(K)$ denotes the number of lattice points in the interior of K. Then*

a. $w(0) = \sqrt{2}$.
b. $w(1) = 3\sqrt{2}/2$.
c. $w(2) = 2$.
d. $w(3) = 2\sqrt{2}$.
e. $w(4) = 3$.
f. $w(5) = 8\sqrt{5}/5$.
g. $w(6) = 8\sqrt{5}/5$.

h. $w(7) = 4$.
i. $w(8) = 4$.
j. $w(9) = 9\sqrt{5}/5$.
k. $w(10) = 2\sqrt{5}$.
l. $w(g) \leq \sqrt{2(g+1)\sqrt{3}}$.

Note the surprising fact that $w(g)$ is not monotonic, since

$$w(1) > w(2) < w(3).$$

We mention some known generalizations for arbitrary convex bodies in the plane. Let $g(K)$ denote the number of lattice points in the interior of the convex body K. Let $w(K)$ denote the width of the body K.

Result 3.17. *If a is a positive real number, then we define*

$$W(a) = \min\{g(K) | w(K) > a\}.$$

Then

a.
$$W\left(\frac{2+\sqrt{3}}{2}\right) = 1.$$

b.
$$\left\lfloor \frac{2a}{2+\sqrt{3}} \right\rfloor^2 \leq W(a) \leq \left\lfloor \frac{a^2}{\sqrt{3}} \right\rfloor.$$

These results are due to Scott [6] and Elkington and Hammer [3].

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