### A Theorem about Collinear Lattice Points

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Abstract. Let S be a set of  $m^n + 1$  lattice points in  $E^n$ . Then either some two points of S span a hole (have a lattice point not in S between them), or some m + 1 points of S are collinear.

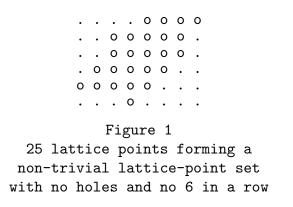
A lattice point is a point in  $E^n$  with integer coordinates. The set of all lattice points in  $E^n$  is denoted by  $Z^n$ . In this note, we will look at some results that show when there must be *m* collinear lattice points in a collection of lattice points in  $Z^n$ .

**Definition.** Two lattice points, x, and y, are said to span a hole in a set S if there is some lattice point between x and y that is not in S. A set of lattice points, S, contains a hole, if some two points of S span a hole.

We now prove the following Ramsey-like theorem: (For other Ramsey-like theorems in  $E^n$ , see section 5.6 of [1] or section 21 of [2].)

**Theorem 1.** Let S be a set of  $m^n + 1$  lattice points in  $E^n$ . Then either some two points of S span a hole, or some m + 1 points of S are collinear.

First note that the set S can be a rather complicated looking set. An example is shown in figure 1 consisting of 25 lattice points in the plane that form a set with no holes and no 6 lattice points in a row. Adding any 26th lattice point, however, (without adding any holes) will force some 6 lattice points to be collinear.



**Proof.** Consider the coordinates of the points modulo m. Since there are only  $m^n$  distinct ordered *n*-tuples of integers modulo m, some two of these must be congruent (mod m). Suppose the two points have coordinates  $(x_1, x_2, \ldots, x_n)$  and  $(x'_1, x'_2, \ldots, x'_n)$ . Then  $x_i \equiv x'_i \pmod{m}$  for  $i = 1, 2, \ldots, n$ . Now consider the points

$$(x_1 + \frac{x_1' - x_1}{m}k, x_2 + \frac{x_2' - x_2}{m}k, \dots, x_n + \frac{x_n' - x_n}{m}k)$$

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as k varies from 0 to m. This is a set of m + 1 collinear points. Furthermore, each point is a lattice point, since  $m|(x'_i - x_i)$  for all i by the congruence condition. Finally, all the m + 1 points belong to S since the first and last ones do and S contains no holes.

Note that the above proof actually gives an effective (and even efficient) procedure for finding the m + 1 collinear lattice points; it is not merely an existence proof.

We note that the quantity  $m^n + 1$  is best possible in the above theorem, for we can always find  $m^n$  lattice points with no holes in which no m + 1 are collinear. Namely, take the  $m^n$  lattice points inside and on the *n*-cube with *m* lattice points along each edge.

Theorem 1 can be rephrased in a number of ways.

**Definition.** A set, S, of lattice points is 2-convex, if it does not contain a hole.

**Proposition 1a.** Let S be a set of  $m^n + 1$  lattice points in  $E^n$  that is 2-convex. Then S must contain some m + 1 lattice points that are collinear.

**Definition.** A set, S, of lattice points is *lattice-convex*, if any lattice point in the convex hull of S is also in S.

The concept of lattice-convexity differs from 2-convexity as can be seen by figure 2 which shows that 2-convexity does not imply lattice-convexity.

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#### Figure 2

A set that is 2-convex but is not lattice convex

However, if x and y are two lattice points in a lattice-convex set S, then any lattice point between x and y must also be a member of S. Thus lattice-convexity implies 2-convexity and we may reformulate Theorem 1 as follows:

**Proposition 1b.** Let S be a set of  $m^n + 1$  lattice points in  $E^n$  that is lattice-convex. Then S must contain some m + 1 lattice points that are collinear.

We can view lattice points in  $E^n$  as vectors emenating from the origin. Such vectors are called lattice vectors.

**Proposition 1c.** Let S be a set of  $m^n + 1$  lattice vectors in  $E^n$ . Then either there is a lattice vector, not in S, that is a convex linear combination of two lattice vectors in S or else some m + 1 vectors in S form an arithmetic progression.

This formulation of Theorem 1 follows from the observation that if m+1 vectors form an arithmetic progression, then their endpoints are collinear.

We can also view Theorem 1 in the light of lattice points inside convex bodies.

**Proposition 1d.** Let K be a convex body in  $E^n$  containing at least  $m^n + 1$  lattice points. Then some m + 1 of these lattice points must be collinear.

This formulation of the theorem follows immediately from the observation that the set of lattice points inside a convex body forms a lattice-convex set.

# Acknowledgement.

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## References

- 1. Ronald L. Graham, Bruce L. Rothschild and Joel H. Spencer, "Ramsey Theory", John Wiley and Sons, New York: 1980.
- 2. J. Hammer, "Unsolved Problems Concerning Lattice Points", Pitman, London: 1977.